Chapter 2. Inverse Trigonometric Functions

1 Mark Questions

1. If $\sin\left(\sin^{-1}\frac{1}{5}+\cos^{-1}x\right)=1$, then find the value of x. **Delhi 2014**

Given, $\sin\left(\sin^{-1}\frac{1}{5}+\cos^{-1}x\right)=1$

$$\Rightarrow \quad \sin^{-1}\frac{1}{5}+\cos^{-1}x=\sin^{-1}(1)$$

$$[\because \sin\theta = x \Rightarrow \theta = \sin^{-1}x]$$

$$\Rightarrow \sin^{-1}\frac{1}{5}+\cos^{-1}x=\sin^{-1}\left(\sin\frac{\pi}{2}\right)\left[\because \sin\frac{\pi}{2}=1\right]$$

$$\Rightarrow \quad \sin^{-1}\frac{1}{5}+\cos^{-1}x=\frac{\pi}{2}$$ **(1/2)**

But we know that, $\sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}$,

$x\in[-1,1]$

$$\therefore \quad \sin^{-1}\frac{1}{5}=\sin^{-1}x \Rightarrow x=\frac{1}{5}$$ **(1/2)**

2. If $\tan^{-1}x+\tan^{-1}y=\frac{\pi}{4}$; $xy<1$, then write the value of $x+y+xy$. **All India 2014**

Given, $\tan^{-1}x + \tan^{-1}y = \frac{\pi}{4}, xy < 1$

We know that,

$$\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left[\frac{x+y}{1-xy}\right], xy < 1$$

$$\therefore \tan^{-1}\left[\frac{x+y}{1-xy}\right] = \frac{\pi}{4} \Rightarrow \frac{x+y}{1-xy} = \tan\frac{\pi}{4} \quad \textbf{(1/2)}$$

$$\Rightarrow \quad x + y = 1 - xy \quad \left[\because \tan\frac{\pi}{4} = 1\right]$$

$$\Rightarrow \quad x + y + xy = 1 \quad \textbf{(1/2)}$$

3. Write the value of $\cos^{-1}\left(-\frac{1}{2}\right) + 2\sin^{-1}\left(\frac{1}{2}\right)$.

Foreign 2014

We have, $\cos^{-1}\left(\frac{-1}{2}\right) + 2\sin^{-1}\left(\frac{1}{2}\right)$

$$= \left[\pi - \cos^{-1}\left(\frac{1}{2}\right)\right] + 2\sin^{-1}\left(\frac{1}{2}\right)$$

$$[\because \cos^{-1}(-x) = \pi - \cos^{-1}x]$$

$$= \left[\pi - \cos^{-1}\left(\cos\frac{\pi}{3}\right)\right] + 2\sin^{-1}\left(\sin\frac{\pi}{6}\right) \quad \textbf{(1/2)}$$

[$\because$ principal value branch of $\cos^{-1}x$ is $[0, \pi]$ and that of $\sin^{-1}x$ is $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$]

$$= \left[\pi - \frac{\pi}{3}\right] + 2 \times \frac{\pi}{6}$$

$$= \frac{2\pi}{3} + \frac{2\pi}{6} = \frac{4\pi + 2\pi}{6} = \pi \quad \textbf{(1/2)}$$

4. Write the principal value of $\cos^{-1}[\cos(680)^\circ]$.

Delhi 2014C

> Firstly, write 680° in the form of $(2\pi-\theta)$ and then use $\cos(2\pi-\theta)=\cos\theta$ and $\cos^{-1}(\cos\theta)=\theta$.

We have, $\cos^{-1}[\cos(680)^\circ]$

$= \cos^{-1}[\cos(2\times 360 - 40)^\circ]$

$= \cos^{-1}[\cos 40^\circ]$ $\quad [\because \cos(2\pi-\theta)=\cos\theta]$

$= 40^\circ$ $\quad [\because \cos^{-1}(\cos\theta)=\theta]$ **(1)**

5. Write the principal value of $\tan^{-1}\left[\sin\left(\frac{-\pi}{2}\right)\right]$.

All India 2014C

We have, $\tan^{-1}\left[\sin\left(-\frac{\pi}{2}\right)\right]$

$= \tan^{-1}[-1]$ $\quad \left[\because \sin\left(\frac{-\pi}{2}\right)=-1\right]$

$\because$ Principal value branch of $\tan^{-1}$ is $\left(\frac{-\pi}{2},\frac{\pi}{2}\right)$.

$\therefore \tan^{-1}(-1)=\tan^{-1}\left(-\tan\frac{\pi}{4}\right)$ $\quad \left[\because \tan\frac{\pi}{4}=1\right]$

$= \tan^{-1}\left[\tan\left(\frac{-\pi}{4}\right)\right]=\frac{-\pi}{4}$ **(1)** $[\tan^{-1}(\tan\theta)=\theta]$

6. Find the value of the following:

$\cot\left(\frac{\pi}{2} - 2\cot^{-1}\sqrt{3}\right)$ **All India 2014 C**

> Firstly, use the property $\cot\left(\frac{\pi}{2} - \theta\right) = \tan\theta$, then put $\cot^{-1}\sqrt{3} = \frac{\pi}{6}$ and simplify.

We have, $\cot\left[\frac{\pi}{2} - 2\cot^{-1}\sqrt{3}\right]$

$= \tan(2\cot^{-1}\sqrt{3}) \left[\because \cot\left(\frac{\pi}{2} - \theta\right) = \tan\theta\right]$ **(1/2)**

$= \tan\left(2 \times \frac{\pi}{6}\right) \left[\because \cot^{-1}\sqrt{3} = \cot^{-1}\left(\cot\frac{\pi}{6}\right) = \frac{\pi}{6}\right]$

$= \tan\left(\frac{\pi}{3}\right) = \sqrt{3}$ **(1/2)**

7. Write the principal value of

$\left[\cos^{-1}\frac{\sqrt{3}}{2} + \cos^{-1}\left(-\frac{1}{2}\right)\right].$ **Delhi 2013C**

We have, $\cos^{-1}\frac{\sqrt{3}}{2} + \cos^{-1}\left(-\frac{1}{2}\right)$

$= \cos^{-1}\frac{\sqrt{3}}{2} + \left[\pi - \cos^{-1}\left(\frac{1}{2}\right)\right]$

$[\because \cos^{-1}(-x) = \pi - \cos^{-1}x]$

$= \frac{\pi}{6} + \pi - \frac{\pi}{3} = \frac{\pi + 6\pi - 2\pi}{6} = \frac{5\pi}{6}$ **(1)**

8. Write the value of $\tan^{-1}\left(\frac{a}{b}\right)-\tan^{-1}\left(\frac{a-b}{a+b}\right)$.

Delhi 2013C

$$\tan^{-1}\left(\frac{a}{b}\right)-\tan^{-1}\left(\frac{a-b}{a+b}\right)=\tan^{-1}\frac{\left[\frac{a}{b}-\left(\frac{a-b}{a+b}\right)\right]}{1+\frac{a}{b}\left(\frac{a-b}{a+b}\right)}$$

$$\left[\because \tan^{-1}A-\tan^{-1}B=\tan^{-1}\left(\frac{A-B}{1+AB}\right)\right]$$

$$=\tan^{-1}\left(\frac{a^2+ab-ab+b^2}{ab+b^2+a^2-ab}\right) \quad \textbf{(1/2)}$$

$$=\tan^{-1}\left(\frac{a^2+b^2}{a^2+b^2}\right)=\tan^{-1}1$$

$$=\tan^{-1}\left(\tan\frac{\pi}{4}\right)=\frac{\pi}{4} \quad [\because \tan^{-1}(\tan\theta)=\theta] \quad \textbf{(1/2)}$$

9. Write the principal value of

$\tan^{-1}(1)+\cos^{-1}\left(-\frac{1}{2}\right)$. **HOTS; Delhi 2013**

Firstly, we check the given angle in principal value branch. If it is not so, then convert it. After that, use the identity

$\tan^{-1}(\tan\theta) = \theta, \cos^{-1}(\cos\theta) = \theta.$

$$\tan^{-1}(1) + \cos^{-1}\left(-\frac{1}{2}\right)$$

$$= \tan^{-1}\left(\tan\frac{\pi}{4}\right) + \cos^{-1}\left(-\cos\frac{\pi}{3}\right) \quad \textbf{(1/2)}$$

$$= \frac{\pi}{4} + \cos^{-1}\left[\cos\left(\pi - \frac{\pi}{3}\right)\right]$$

[since, principal value branch of $\cos^{-1} x$ is $[0, \pi]$, so we convert $-\cos\theta = \cos(\pi - \theta)$]

$$= \frac{\pi}{4} + \cos^{-1}\left[\cos\frac{2\pi}{3}\right]$$

$$= \frac{\pi}{4} + \frac{2\pi}{3} = \frac{3\pi + 8\pi}{12}$$

$$= \frac{11\pi}{12} \quad \textbf{(1/2)}$$

10. Write the value of $\tan\left(2\tan^{-1}\frac{1}{5}\right)$. **Delhi 2013**

$$\tan\left(2\tan^{-1}\frac{1}{5}\right) = \tan\left[\tan^{-1}\left(\frac{2 \times \frac{1}{5}}{1 - \left(\frac{1}{5}\right)^2}\right)\right] \quad \textbf{(1/2)}$$

$$\left[\because 2\tan^{-1} x = \tan^{-1}\left(\frac{2x}{1 - x^2}\right)\right]$$

$$= \tan\left[\tan^{-1}\left(\frac{2 \times 5}{24}\right)\right] = \tan\left[\tan^{-1}\left(\frac{5}{12}\right)\right] = \frac{5}{12}$$

$[\because \tan(\tan^{-1}\theta) = \theta]$ **(1/2)**

11. Write the value of $\tan^{-1}\left[2\sin\left(2\cos^{-1}\frac{\sqrt{3}}{2}\right)\right]$.

All India 2013

$$\tan^{-1}\left[2\sin\left(2\cos^{-1}\frac{\sqrt{3}}{2}\right)\right]$$

$$= \tan^{-1}\left[2\sin\left\{\cos^{-1}\left(2\cdot\frac{3}{4}-1\right)\right\}\right]$$

$[\because 2\cos^{-1}x = \cos^{-1}(2x^2-1)]$

$$= \tan^{-1}\left[2\sin\left\{\cos^{-1}\left(\frac{3}{2}-1\right)\right\}\right]$$

$$= \tan^{-1}\left[2\sin\left\{\cos^{-1}\left(\frac{1}{2}\right)\right\}\right] \quad \textbf{(1/2)}$$

$$= \tan^{-1}\left[2\sin\left\{\cos^{-1}\cdot\left(\cos\frac{\pi}{3}\right)\right\}\right]$$

$$= \tan^{-1}\left[2\sin\frac{\pi}{3}\right] [\because \cos^{-1}(\cos\theta) = \theta]$$

$$= \tan^{-1}\left(2\cdot\frac{\sqrt{3}}{2}\right)$$

$$= \tan^{-1}(\sqrt{3}) = \tan^{-1}\left(\tan\frac{\pi}{3}\right) = \frac{\pi}{3}$$

$[\because \tan^{-1}(\tan\theta) = \theta]$ **(1/2)**

12. Write the principal value of

$\tan^{-1}(\sqrt{3}) - \cot^{-1}(-\sqrt{3})$. All India 2013

$\tan^{-1}(\sqrt{3}) - \cot^{-1}(-\sqrt{3})$

$= \tan^{-1}(\sqrt{3}) - \{\pi - \cot^{-1}(\sqrt{3})\}$

$\left[\begin{array}{l}\because \text{principal value branch of } \cot^{-1}x \text{ is } (0, \pi.) \\ \therefore\ \cot^{-1}(-x) = \pi - \cot^{-1}x\end{array}\right]$

$= \tan^{-1}\sqrt{3} - \pi + \cot^{-1}\sqrt{3}$

$= (\tan^{-1}\sqrt{3} + \cot^{-1}\sqrt{3}) - \pi$

$= \frac{\pi}{2} - \pi = -\frac{\pi}{2} \left[\because \tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}\right]$ **(1)**

13. Write the value of $\cos^{-1}\left(\frac{1}{2}\right) - 2\sin^{-1}\left(-\frac{1}{2}\right)$.

Delhi 2012

> Firstly, we check the given angle in principal value branch, then use the identity $\sin^{-1}(\sin\theta) = \theta$ and $\cos(\cos^{-1}\theta) = \theta$.

$\cos^{-1}\left(\frac{1}{2}\right) - 2\sin^{-1}\left(-\frac{1}{2}\right)$

$= \cos^{-1}\left(\cos\frac{\pi}{3}\right) - 2\sin^{-1}\left[\sin\left(-\frac{\pi}{6}\right)\right]$

$\left[\because \text{principal value branch of } \cos^{-1}x\right.$

$\left.\text{is } [0, \pi] \text{ and that of } \sin^{-1} \text{ is } \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\right]$

$= \frac{\pi}{3} - 2\left(-\frac{\pi}{6}\right)$

$[\because \cos^{-1}(\cos\theta) = \theta \text{ and } \sin^{-1}(\sin\theta) = \theta]$

$= \frac{\pi}{3} + \frac{\pi}{3} = \frac{2\pi}{3}$ **(1)**

14. Find the principal value of
$\tan^{-1}\sqrt{3} - \sec^{-1}(-2)$.

All India 2012

Given expression is not standard identity, so we separately find the value of $\tan^{-1}(\sqrt{3})$ and $\sec^{-1}(-2)$, then simplify it.

We know that, the principal value branch of $\tan^{-1}x$ is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and that of $\sec^{-1}$ is $[0, \pi] - \left\{\frac{\pi}{2}\right\}$.

$\therefore \tan^{-1}\sqrt{3} - \sec^{-1}(-2)$

$$= \tan^{-1}\left(\tan\frac{\pi}{3}\right) - \sec^{-1}\left(\sec\frac{2\pi}{3}\right)$$

$$\left[\because \tan\frac{\pi}{3} = \sqrt{3} \text{ and } \sec\frac{2\pi}{3} = -2\right]$$

$$= \frac{\pi}{3} - \frac{2\pi}{3} = \frac{-\pi}{3}$$

$[\because \tan^{-1}(\tan\theta) = \theta$ and $\sec^{-1}(\sec\theta) = \theta]$ **(1)**

15. Using the principal values, write the value of $\cos^{-1}\left(\frac{1}{2}\right) + 2\sin^{-1}\left(\frac{1}{2}\right)$.

All India 2012C

We know that, the principal value branch of $\cos^{-1}x$ is $[0, \pi]$ and of $\sin^{-1}x$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

$\therefore \cos^{-1}\left(\frac{1}{2}\right) + 2\sin^{-1}\left(\frac{1}{2}\right)$

$$= \cos^{-1}\left(\cos\frac{\pi}{3}\right) + 2\sin^{-1}\left(\sin\frac{\pi}{6}\right)$$

$$\left[\because \cos\frac{\pi}{3} = \frac{1}{2} \text{ and } \sin\frac{\pi}{6} = \frac{1}{2}\right]$$

$$= \frac{\pi}{3} + \frac{\pi}{3} = \frac{2\pi}{3}$$

$[\because \cos^{-1}(\cos\theta) = \theta$ and $\sin^{-1}(\sin\theta) = \theta]$ **(1)**

16. Write the value of $\sin\left[\frac{\pi}{3} - \sin^{-1}\left(-\frac{1}{2}\right)\right]$.

Delhi 2011

Firstly, find the principal value of $\sin^{-1}\left(\frac{1}{2}\right)$, then solve it.

$$\sin\left[\frac{\pi}{3} - \sin^{-1}\left(-\frac{1}{2}\right)\right] = \sin\left[\frac{\pi}{3} + \sin^{-1}\left(\frac{1}{2}\right)\right]$$

$[\because \sin^{-1}(-\theta) = -\sin^{-1}\theta]$

$$= \sin\left[\frac{\pi}{3} + \sin^{-1}\left(\sin\frac{\pi}{6}\right)\right] \quad \left[\because \sin\frac{\pi}{6} = \frac{1}{2}\right]$$

$$= \sin\left[\frac{\pi}{3} + \frac{\pi}{6}\right] = \sin\frac{\pi}{2} = 1$$

$[\because \sin^{-1}(\sin\theta) = \theta]$ **(1)**

NOTE Please be careful that we do not write $\sin^{-1}(-\sin\theta) = \theta$.

17. Write the value of $\tan^{-1}\left(\tan\frac{3\pi}{4}\right)$.

HOTS; Delhi 2011

Firstly, we check the given angle in principal value. If it is so, then use the identity $\tan^{-1}(\tan\theta)=\theta$.

We know that, principal value branch of $\tan^{-1}x$ is $\left(\frac{-\pi}{2},\frac{\pi}{2}\right)$.

$\therefore$ Principal value of $\tan^{-1}\left(\tan\frac{3\pi}{4}\right)$

$$=\tan^{-1}\left[\tan\left(\pi-\frac{\pi}{4}\right)\right]$$

$$\left[\begin{array}{l}\because \frac{3\pi}{4}\notin\left(\frac{-\pi}{2},\frac{\pi}{2}\right)\\ \text{so, write } \frac{3\pi}{4} \text{ as} \left(\pi-\frac{\pi}{4}\right)\end{array}\right]$$

$$=\tan^{-1}\left(-\tan\frac{\pi}{4}\right) \quad [\because \tan(\pi-\theta)=-\tan\theta]$$

$$=\tan^{-1}\left[\tan\left(-\frac{\pi}{4}\right)\right] \quad [\because (-\tan\theta)=\tan(-\theta)]$$

$$=-\frac{\pi}{4}\in\left(\frac{-\pi}{2},\frac{\pi}{2}\right)$$

$$\therefore \quad \tan^{-1}\left(\tan\frac{3\pi}{4}\right)=-\frac{\pi}{4} \quad \textbf{(1)}$$

NOTE Please be careful, we do not write

$\tan^{-1}\left(\tan\frac{3\pi}{4}\right)=\frac{3\pi}{4}$, because $\frac{3\pi}{4}\notin\left(\frac{-\pi}{2},\frac{\pi}{2}\right)$.

18. Write the value of $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)$.

HOTS; Delhi 2011, 2009; All India 2009

We know that, the principal value branch of $\cos^{-1} x$ is $[0, \pi]$.

$$\therefore \quad \cos^{-1}\left(\cos\frac{7\pi}{6}\right) = \cos^{-1}\left[\cos\left(2\pi - \frac{5\pi}{6}\right)\right]$$

$$\left[\because \frac{7\pi}{6} \notin [0, \pi], \text{ so write } \frac{7\pi}{6} = \left(2\pi - \frac{5\pi}{6}\right)\right]$$

$$= \cos^{-1}\left[\cos\frac{5\pi}{6}\right] \quad \left[\because \cos(2\pi - \theta) = \cos\theta \text{ and } \frac{5\pi}{6} \in [0, \pi]\right]$$

$$= \frac{5\pi}{6}$$

$$\therefore \quad \cos^{-1}\left(\cos\frac{7\pi}{6}\right) = \frac{5\pi}{6}$$ **(1)**

19. What is the principal value of $\cos^{-1}\left(\cos\frac{2\pi}{3}\right) + \sin^{-1}\left(\sin\frac{2\pi}{3}\right)$?

All India 2011, 2008, 2009C

We know that, the principal value branch of $\cos^{-1} x$ is $[0, \pi]$ and for $\sin^{-1} x$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

$$\therefore \cos^{-1}\left(\cos \frac{2\pi}{3}\right) + \sin^{-1}\left(\sin \frac{2\pi}{3}\right)$$

$$= \frac{2\pi}{3} + \sin^{-1}\left[\sin\left(\pi - \frac{\pi}{3}\right)\right]$$

$$\left[\because \frac{2\pi}{3} \notin \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], \text{ so write } \frac{2\pi}{3} \text{ as } \left(\pi - \frac{\pi}{3}\right)\right]$$

$$= \frac{2\pi}{3} + \sin^{-1}\left(\sin \frac{\pi}{3}\right) \quad [\because \sin(\pi - \theta) = \sin\theta]$$

$$= \frac{2\pi}{3} + \frac{\pi}{3} = \frac{3\pi}{3} = \pi \quad [\because \sin^{-1}(\sin\theta) = \theta] \quad \textbf{(1)}$$

20. What is the principal value of $\tan^{-1}(-1)$?

Foreign 2011, 2008C

We know that, the principal value branch of $\tan^{-1} x$ is $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$.

$$\therefore \tan^{-1}(-1) = \tan^{-1}\left(-\tan \frac{\pi}{4}\right) \quad \left[\because \tan \frac{\pi}{4} = 1\right]$$

$$= \tan^{-1}\left[\tan\left(-\frac{\pi}{4}\right)\right] \quad [\because (-\tan\theta) = \tan(-\theta)]$$

$$= -\frac{\pi}{4} \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$$

Hence, $\tan^{-1}(-1) = -\frac{\pi}{4}$ **(1)**

21. Using the principal values, write the value of $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$.

HOTS; All India 2011C

We know that, the principal value branch of $\sin^{-1} x$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

$$\therefore \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \sin^{-1}\left(-\sin\frac{\pi}{3}\right)$$

$$= \sin^{-1}\left[\sin\left(-\frac{\pi}{3}\right)\right] \quad [\because \sin(-\theta) = -\sin\theta]$$

$$= -\frac{\pi}{3} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \quad [\because \sin^{-1}(\sin\theta) = \theta]$$

Hence, $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3}$ **(1)**

22. Write the principal value of $\sin^{-1}\left(-\frac{1}{2}\right)$.

Delhi 2011C

We know that, the principal value branch of $\sin^{-1} x$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

$$\therefore \sin^{-1}\left(-\frac{1}{2}\right) = \sin^{-1}\left(-\sin\frac{\pi}{6}\right)$$

$$= \sin^{-1}\left[\sin\left(-\frac{\pi}{6}\right)\right] \quad [\because \sin(-\theta) = -\sin\theta]$$

$$= -\frac{\pi}{6} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \quad [\because \sin^{-1}(\sin\theta) = \theta]$$

Hence, $\sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$ **(1)**

23. Write the principal value of $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$.

Delhi 2010

We know that, the principal value branch of

$\sin^{-1} x$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

$$\therefore \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \sin^{-1}\left(\sin\frac{\pi}{3}\right) \left[\because \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}\right]$$

$$= \frac{\pi}{3} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$[\because \sin^{-1}(\sin\theta) = \theta]$ **(1)**

24. What is the principal value of $\sec^{-1}(-2)$?

All India 2010

We know that, the principal value branch of $\sec^{-1}$ is $[0, \pi] - \{\pi/2\}$.

$$\therefore \sec^{-1}(-2) = \sec^{-1}\left(-\sec\frac{\pi}{3}\right)$$

$\Big[$here, $\sec^{-1}(-\sec\frac{\pi}{3}) \neq -\frac{\pi}{3}$

since, $\frac{-\pi}{3} \notin [0, \pi] - \left\{\frac{\pi}{2}\right\}\Big]$

$$= \sec^{-1}\left[\sec\left(\pi - \frac{\pi}{3}\right)\right] \quad [\because \sec(\pi - \theta) = -\sec\theta]$$

$$= \sec^{-1}\left(\sec\frac{2\pi}{3}\right) = \frac{2\pi}{3} \in [0, \pi] - \left\{\frac{\pi}{2}\right\}$$

$[\because \sec^{-1}(\sec\theta) = \theta]$

Hence, $\sec^{-1}(-2) = \frac{2\pi}{3}$ **(1)**

25. What is the domain of the function $\sin^{-1} x$?

Foreign 2010

The domain of $\sin^{-1} x$ is $-1 \leq x \leq 1$. **(1)**

26. Using the principal values, find the value of $\cos^{-1}\left(\cos\frac{13\pi}{6}\right)$. **All India 2010C**

As the principal value branch of $\cos^{-1} x$ is $[0, \pi]$.

$$\therefore \quad \cos^{-1}\left(\cos\frac{13\pi}{6}\right) \neq \frac{13\pi}{6} \text{ as } \frac{13\pi}{6} \notin [0, \pi]$$

Now, $\cos^{-1}\left(\cos\frac{13\pi}{6}\right) = \cos^{-1}\left[\cos\left(2\pi + \frac{\pi}{6}\right)\right]$

$$= \cos^{-1}\left(\cos\frac{\pi}{6}\right) \quad [\because \cos(2\pi + \theta) = \cos\theta]$$

$$= \frac{\pi}{6} \in [0, \pi] \quad [\because \cos^{-1}(\cos\theta) = \theta]$$

Hence, $\cos^{-1}\left(\cos\frac{13\pi}{6}\right) = \frac{\pi}{6}$ **(1)**

27. If $\tan^{-1}(\sqrt{3}) + \cot^{-1} x = \frac{\pi}{2}$, then find the value of x. **All India 2010C**

Given, $\tan^{-1}\sqrt{3} + \cot^{-1} x = \frac{\pi}{2}$

$$\Rightarrow \tan^{-1}\sqrt{3} = \frac{\pi}{2} - \cot^{-1} x$$

$$\Rightarrow \tan^{-1}\sqrt{3} = \tan^{-1} x \quad \left[\because \tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}\right]$$

On equating both sides, we get

$$x = \sqrt{3}$$ **(1)**

28. Write the principal value of $\sin^{-1}\left(\sin\frac{3\pi}{5}\right)$. **Delhi 2009**

As we know that, the principal value branch of $\sin^{-1} x$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

$$\therefore \sin^{-1}\left[\sin\left(\frac{3\pi}{5}\right)\right] \neq \frac{3\pi}{5} \quad \left[\because \frac{3\pi}{5} \notin \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\right]$$

Now, $\sin^{-1}\left(\sin \frac{3\pi}{5}\right)$

$$= \sin^{-1}\left[\sin\left(\pi - \frac{2\pi}{5}\right)\right]$$

$$= \sin^{-1}\left(\sin \frac{2\pi}{5}\right) \quad [\because \sin(\pi - \theta) = \sin \theta]$$

$$= \frac{2\pi}{5} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \quad [\because \sin^{-1}(\sin \theta) = \theta]$$

Hence, $\sin^{-1}\left(\sin \frac{3\pi}{5}\right) = \frac{2\pi}{5}$ **(1)**

29. Using the principal values, evaluate $\tan^{-1}(1) + \sin^{-1}\left(-\frac{1}{2}\right)$.

Delhi 2009C

We know that, the principal value of $\tan^{-1} x$ is $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ and that of $\sin^{-1}$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

$\therefore \tan^{-1}(1) + \sin^{-1}\left(-\frac{1}{2}\right)$

$$= \tan^{-1}\left(\tan\frac{\pi}{4}\right) + \sin^{-1}\left(-\sin\frac{\pi}{6}\right)$$

$$\left[\because \tan\frac{\pi}{4} = 1 \text{ and } \sin\frac{\pi}{6} = \frac{1}{2}\right]$$

$$= \tan^{-1}\left(\tan\frac{\pi}{4}\right) + \sin^{-1}\left[\sin\left(-\frac{\pi}{6}\right)\right]$$

$$[\because \sin(-\theta) = -\sin\theta]$$

$$= \frac{\pi}{4} - \frac{\pi}{6} = \frac{\pi}{12}$$

$[\because \tan^{-1}(\tan\theta) = \theta \text{ and } \sin^{-1}(\sin\theta) = \theta]$ **(1)**

30. Find the principal value of $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$.

All India 2008C

$$\cos^{-1}\frac{\sqrt{3}}{2} = \cos^{-1}\left(\cos\frac{\pi}{6}\right) \quad \left[\because \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}\right]$$

Since, $\frac{\pi}{6} \in [0, \pi]$

{As principal value branch of $\cos^{-1} x$ is $[0, \pi]$}

$\therefore$

$$\cos^{-1}\frac{\sqrt{3}}{2} = \frac{\pi}{6}$$ **(1)**

6 Mark Questions

31. Solve the following equation:

$$\cos(\tan^{-1} x) = \sin\left(\cot^{-1}\frac{3}{4}\right)$$

Foreign 2014; All India 2013;HOTS

Given, $\cos(\tan^{-1} x) = \sin\left(\cot^{-1} \frac{3}{4}\right)$

$$\Rightarrow \quad \sin\left\{\frac{\pi}{2} - \tan^{-1} x\right\} = \sin\left(\cot^{-1} \frac{3}{4}\right)$$

$$\left[\because \sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta\right] \quad \textbf{(1)}$$

On equating both sides, we get

$$\frac{\pi}{2} - \tan^{-1} x = \cot^{-1} \frac{3}{4} \quad \textbf{(1)}$$

$$\Rightarrow \quad \tan^{-1} x + \cot^{-1} \frac{3}{4} = \frac{\pi}{2} \quad \textbf{(1)}$$

It is only possible, when $x = \frac{3}{4}$.

$$\left[\because \tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}, x \in R\right] \quad \textbf{(1)}$$

32. Solve following equation for x.

$$\tan^{-1}\left(\frac{1 - x}{1 + x}\right) = \frac{1}{2} \tan^{-1} x$$

Foreign 2011C, 08C; All India 2014C, 2010, 2009C

Multiply both sides by 2 and then in LHS, use the relation $2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$ and solve it.Then, equate both sides to get the value of x.

Given, $\tan^{-1}\left(\frac{1-x}{1+x}\right) = \frac{1}{2}\tan^{-1}x,\ x > 0$

$$\Rightarrow \quad 2\tan^{-1}\left(\frac{1-x}{1+x}\right) = \tan^{-1}x$$

$$\Rightarrow \quad \tan^{-1}\left[\frac{2\left(\frac{1-x}{1+x}\right)}{1-\left(\frac{1-x}{1+x}\right)^2}\right] = \tan^{-1}x \qquad \textbf{(1½)}$$

$$\left[\because 2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)\right]$$

$$\Rightarrow \quad \tan^{-1}\left[\frac{2(1-x)(1+x)}{(1+x)^2-(1-x)^2}\right] = \tan^{-1}x$$

$$\Rightarrow \quad \tan^{-1}\left(\frac{2-2x^2}{4x}\right) = \tan^{-1}x$$

$$[\because a^2 - b^2 = (a-b)(a+b)]$$

$$\Rightarrow \quad \frac{1-x^2}{2x} = x \quad \Rightarrow \quad 1-x^2 = 2x^2$$

$$\Rightarrow \quad 3x^2 = 1 \Rightarrow x^2 = \frac{1}{3} \Rightarrow x = \pm\frac{1}{\sqrt{3}} \ \textbf{(1½)}$$

But given, $x > 0$

$$\therefore \quad x = \frac{1}{\sqrt{3}} \qquad \textbf{(1)}$$

33. Solve for x, $2\tan^{-1}(\cos x) = \tan^{-1}(2\,\text{cosec}\ x)$.

Delhi 2014C; All India 2009

Given equation is

$$2\tan^{-1}(\cos x) = \tan^{-1}(2\operatorname{cosec} x)$$

$$\Rightarrow \quad \tan^{-1}\left(\frac{2\cos x}{1-\cos^2 x}\right) = \tan^{-1}\left(\frac{2}{\sin x}\right)$$

$$\left[\because 2\tan^{-1} x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)\right] \quad \textbf{(1½)}$$

$$\Rightarrow \quad \frac{2\cos x}{\sin^2 x} = \frac{2}{\sin x} \; [\because \sin^2 x = 1-\cos^2 x] \ldots \text{(i)}$$

$$\Rightarrow \quad \sin x \cdot \cos x - \sin^2 x = 0$$

$$\Rightarrow \quad \sin x(\cos x - \sin x) = 0$$

$$\Rightarrow \quad \sin x = 0 \text{ or } \cos x = \sin x$$

$$\Rightarrow \quad \sin x = \sin 0 \text{ or } \cot x = 1 = \cot \pi/4$$

$$\Rightarrow \quad x = 0 \text{ or } \frac{\pi}{4} \quad \textbf{(1½)}$$

But here at $x = 0$, the given equation does not exist. Hence, $x = \frac{\pi}{4}$ is the only solution. **(1)**

34. Solve for x, $\tan^{-1} x + 2\cot^{-1} x = \frac{2\pi}{3}$.

All India 2014C;Delhi 2009C

The given equation is $\tan^{-1} x + 2\cot^{-1} x = \frac{2\pi}{3}$

We know that, $\cot^{-1} x = \tan^{-1}\frac{1}{x}$

So, the given equation can be written as

$$\tan^{-1} x + 2\tan^{-1}\left(\frac{1}{x}\right) = \frac{2\pi}{3}$$

$$\Rightarrow \quad \tan^{-1} x + \tan^{-1}\left(\frac{2 \times \frac{1}{x}}{1-\frac{1}{x^2}}\right) = \frac{2\pi}{3}$$

$$\left[\because 2\tan^{-1} x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)\right]$$

$$\Rightarrow \quad \tan^{-1} x + \tan^{-1}\left(\frac{\frac{2}{x}}{\frac{x^2-1}{x^2}}\right) = \frac{2\pi}{3}$$

$$\Rightarrow \quad \tan^{-1} x + \tan^{-1}\left(\frac{2x}{x^2-1}\right) = \frac{2\pi}{3} \qquad \textbf{(1½)}$$

$$\Rightarrow \quad \tan^{-1}\left(\frac{x + \frac{2x}{x^2-1}}{1 - \frac{2x^2}{x^2-1}}\right) = \frac{2\pi}{3}$$

$$\left[\because \tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)\right]$$

$$\Rightarrow \quad \frac{x^3 - x + 2x}{x^2 - 1 - 2x^2} = \tan\frac{2\pi}{3} \qquad \textbf{(1½)}$$

$$\Rightarrow \quad \frac{x^3 + x}{-1 - x^2} = \tan\left(\pi - \frac{\pi}{3}\right)$$

$$\Rightarrow \quad \frac{x^3 + x}{-(1+x^2)} = -\tan\frac{\pi}{3}$$

$$[\because \tan(\pi - \theta) = -\tan\theta]$$

$$\Rightarrow \quad \frac{x(1+x^2)}{-(1+x^2)} = -\sqrt{3} \quad \Rightarrow x = \sqrt{3} \qquad \textbf{(1)}$$

35. Prove that

$$\cot^{-1}\left(\frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}}\right) = \frac{x}{2};\ x \in \left(0, \frac{\pi}{4}\right).$$

Delhi 2014C, 2011; All India 2009

• $\text{LHS} = \cot^{-1}\left[\dfrac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right]$

$$= \cot^{-1}\left[\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}} \times \frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}+\sqrt{1-\sin x}}\right] \quad \textbf{(1)}$$

[multiplying and dividing by conjugate of denominator, i.e. by $\sqrt{1+\sin x}+\sqrt{1-\sin x}$]

$$= \cot^{-1}\left[\frac{(\sqrt{1+\sin x}+\sqrt{1-\sin x})^2}{(\sqrt{1+\sin x})^2-(\sqrt{1-\sin x})^2}\right] \quad \textbf{(1)}$$

$[\because (a+b)(a-b) = a^2 - b^2]$

$$= \cot^{-1}\left[\frac{1+\sin x+1-\sin x+2\sqrt{1-\sin^2 x}}{1+\sin x-1+\sin x}\right]$$

$$= \cot^{-1}\left[\frac{2+2\cos x}{2\sin x}\right] \quad [\because 1-\sin^2 x = \cos^2 x] \ \textbf{(1)}$$

$$= \cot^{-1}\left[\frac{1+\cos x}{\sin x}\right] = \cot^{-1}\left[\frac{2\cos^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}\right]$$

$\left[\because \cos x = 2\cos^2\frac{x}{2}-1 \text{ and } \sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}\right]$

$$= \cot^{-1}\left[\cot\frac{x}{2}\right] = \frac{x}{2} = \text{RHS} \quad \textbf{(1)}$$

Hence proved.

36. Prove that

$$2\tan^{-1}\left(\frac{1}{5}\right)+\sec^{-1}\left(\frac{5\sqrt{2}}{7}\right)+2\tan^{-1}\left(\frac{1}{8}\right)=\frac{\pi}{4}.$$

Delhi 2014

$\therefore$ LHS $= 2\tan^{-1}\left(\frac{1}{5}\right) + \sec^{-1}\left(\frac{5\sqrt{2}}{7}\right) + 2\tan^{-1}\left(\frac{1}{8}\right)$

$$= 2\left\{\tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{8}\right\} + \sec^{-1}\frac{5\sqrt{2}}{7}$$

$$= 2\tan^{-1}\left\{\frac{\frac{1}{5}+\frac{1}{8}}{1-\frac{1}{5}\times\frac{1}{8}}\right\} + \tan^{-1}\sqrt{\left(\frac{5\sqrt{2}}{7}\right)^2 - 1}$$

$$\left[\because \tan^{-1}x + \tan^{-1}y = \tan^{-1}\left\{\frac{x+y}{1-xy}\right\} \text{ and } \sec^{-1}x = \tan^{-1}\sqrt{x^2-1}\right]$$ **(1)**

$$= 2\tan^{-1}\frac{13}{39} + \tan^{-1}\sqrt{\frac{50}{49} - 1}$$

$= 2\tan^{-1}\frac{1}{3} + \tan$ … 7 **(1)**

$$= \tan^{-1}\left\{\frac{2\times\frac{1}{3}}{1-\left(\frac{1}{3}\right)^2}\right\} + \tan^{-1}\frac{1}{7}$$

$$\left[\because 2\tan^{-1}x = \tan^{-1}\left\{\frac{2x}{1-x^2}\right\}\right]$$ **(1)**

$$= \tan^{-1}\frac{3}{4} + \tan^{-1}\frac{1}{7} = \tan^{-1}\left\{\frac{\frac{3}{4}+\frac{1}{7}}{1-\frac{3}{4}\times\frac{1}{7}}\right\}$$

$$= \tan^{-1}(1) = \tan^{-1}\left(\tan\frac{\pi}{4}\right) = \frac{\pi}{4} = \text{RHS}$$ **(1)**

$[\because \tan^{-1}(\tan\theta) = \theta]$ **Hence proved.**

37. Prove that $\tan^{-1}\left[\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right]$

$$= \frac{\pi}{4} - \frac{1}{2}\cos^{-1} x, \quad \frac{-1}{\sqrt{2}} \le x \le 1.$$

All India 2014, 2011,2014C; HOTS

$$\text{LHS} = \tan^{-1}\left[\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right]$$

Let $x = \cos\theta$, so that $\cos^{-1} x = \theta$ **(1)**

$$\text{Then, LHS} = \tan^{-1}\left[\frac{\sqrt{1+\cos\theta}-\sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta}+\sqrt{1-\cos\theta}}\right]$$

$$= \tan^{-1}\left[\frac{\sqrt{2}\cos\frac{\theta}{2}-\sqrt{2}\sin\frac{\theta}{2}}{\sqrt{2}\cos\frac{\theta}{2}+\sqrt{2}\sin\frac{\theta}{2}}\right]$$

$\left[\because 1+\cos\theta = 2\cos^2\frac{\theta}{2} \text{ and } 1-\cos\theta = 2\sin^2\frac{\theta}{2}\right]$ **(1)**

$$= \tan^{-1}\left[\frac{\cos\frac{\theta}{2}-\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}+\sin\frac{\theta}{2}}\right] = \tan^{-1}\left[\frac{1-\tan\frac{\theta}{2}}{1+\tan\frac{\theta}{2}}\right]$$

[dividing numerator and denominator by $\cos(\theta/2)$] **(1)**

$$= \tan^{-1}\left[\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right]$$

$\left[\because \tan(x-y) = \frac{\tan x - \tan y}{1+\tan x \tan y}\right]$

$$= \frac{\pi}{4} - \frac{\theta}{2} \qquad [\because \tan^{-1}(\tan\theta) = \theta]$$

$$= \frac{\pi}{4} - \frac{1}{2}\cos^{-1} x \qquad [\because \theta = \cos^{-1} x]$$

= RHS **Hence proved.**

38. If $\tan^{-1}\left(\frac{x-2}{x-4}\right) + \tan^{-1}\left(\frac{x+2}{x+4}\right) = \frac{\pi}{4}$, then find the value of x. **All India 2014**

Given, $\tan^{-1}\left(\frac{x-2}{x-4}\right)+\tan^{-1}\left(\frac{x+2}{x+4}\right)=\frac{\pi}{4}$

$$\Rightarrow \tan^{-1}\left[\frac{\frac{x-2}{x-4}+\frac{x+2}{x+4}}{1-\left(\frac{x-2}{x-4}\right)\left(\frac{x+2}{x+4}\right)}\right]=\frac{\pi}{4}$$

$$\left[\because \tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right)\right] \quad \textbf{(1)}$$

$$\Rightarrow \tan^{-1}\left[\frac{\frac{(x-2)(x+4)+(x+2)(x-4)}{(x-4)(x+4)}}{\frac{(x-4)(x+4)-(x-2)(x+2)}{(x-4)(x+4)}}\right]=\frac{\pi}{4}$$

$$\Rightarrow \frac{x^2-2x+4x-8+x^2-4x+2x-8}{(x^2-16)-(x^2-4)}$$

$$=\tan\frac{\pi}{4} \quad \textbf{(1)}$$

$$\Rightarrow \frac{2x^2-16}{-12}=1$$

$$\Rightarrow 2x^2-16=-12$$

$$\Rightarrow 2x^2=-12+16$$

$$\Rightarrow 2x^2=4 \Rightarrow x^2=2$$

$$\therefore x=\pm\sqrt{2} \quad \textbf{(1)}$$

Hence, $\sqrt{2}$ and $-\sqrt{2}$ are the required values of x.

(1)

39. Prove that

$$\cos^{-1}(x)+\cos^{-1}\left\{\frac{x}{2}+\frac{\sqrt{3-3x^2}}{2}\right\}=\frac{\pi}{3}.$$

All India 2014C

We have to prove

$$\cos^{-1}(x) + \cos^{-1}\left\{\frac{x}{2} + \frac{\sqrt{3-3x^2}}{2}\right\} = \frac{\pi}{3}$$

$$\text{LHS} = \cos^{-1}(x) + \cos^{-1}\left\{\frac{x}{2} + \frac{\sqrt{3-3x^2}}{2}\right\}$$

Let $\cos^{-1} x = \alpha \Rightarrow x = \cos\alpha$ **(1)**

Then, $\text{LHS} = \alpha + \cos^{-1}$

$$\left[\cos\alpha \cdot \cos\frac{\pi}{3} + \frac{\sqrt{3}}{2}\sqrt{1-\cos^2\alpha}\right]$$

$$= \alpha + \cos^{-1}\left[\cos\frac{\pi}{3}\cos\alpha + \sin\frac{\pi}{3}\sin\alpha\right]$$

$$\left[\because \sin\alpha = \sqrt{1-\cos^2\alpha}, \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}\right]$$ **(1)**

$$= \alpha + \cos^{-1}\left[\cos\left(\frac{\pi}{3} - \alpha\right)\right]$$

$[\because \cos(A-B) = \cos A \cos B + \sin A \sin B]$ **(1)**

$$= \alpha + \frac{\pi}{3} - \alpha = \frac{\pi}{3} = \text{RHS}$$ **(1)**

40. Prove that $\cot^{-1}7 + \cot^{-1}8 + \cot^{-1}18 = \cot^{-1}3$

Foreign 2014

To prove, $\cot^{-1}7+\cot^{-1}8+\cot^{-1}18=\cot^{-1}3$

LHS $=\cot^{-1}7+\cot^{-1}8+\cot^{-1}18$

$$=\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{8}+\tan^{-1}\frac{1}{18}$$

$$\left[\because \cot^{-1}x=\frac{1\cdot}{\tan^{-1}x}\right] \textbf{(1)}$$

$$=\tan^{-1}\left(\frac{\frac{1}{7}+\frac{1}{8}}{1-\frac{1}{7}\times\left(\frac{1}{8}\right)}\right)+\tan^{-1}\frac{1}{18}$$

$$\left[\because \tan^{-1}A+\tan^{-1}B=\tan^{-1}\left(\frac{A+B}{1-AB}\right)\right] \textbf{(1)}$$

$$=\tan^{-1}\left(\frac{15}{55}\right)+\tan^{-1}\frac{1}{18}$$

$$=\tan^{-1}\left(\frac{3}{11}\right)+\tan^{-1}\frac{1}{18}$$

$$=\tan^{-1}\left(\frac{\frac{3}{11}+\frac{1}{18}}{1-\left(\frac{3}{11}\right)\times\left(\frac{1}{18}\right)}\right)=\tan^{-1}\left(\frac{65}{195}\right) \quad \textbf{(1)}$$

$$=\tan^{-1}\left(\frac{1}{3}\right)=\cot^{-1}3=\text{RHS} \quad \textbf{(1)}$$

Hence proved.

41. Prove the following:

$$\cot^{-1}\left[\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right]=\frac{x}{2}\,;x\in\left(0,\frac{\pi}{4}\right)$$

Delhi 2014, 2011; HOTS; All India 2009

Use the relation

$$1+\sin x=\cos^2\frac{x}{2}+\sin^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}$$

$$=\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2$$

For $x\in\left(0,\frac{\pi}{4}\right)$,

$$1-\sin x=\cos^2\frac{x}{2}+\sin^2\frac{x}{2}-2\sin\frac{x}{2}\cos\frac{x}{2}$$

$$=\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2$$ and then simplify.

$$\text{LHS}=\cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right)$$

$$=\cot^{-1}\left[\frac{\sqrt{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2}+\sqrt{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2}}{\sqrt{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2}-\sqrt{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2}}\right]$$

(1½)

$$\text{LHS}=\cot^{-1}\left(\frac{\cos\frac{x}{2}+\sin\frac{x}{2}+\cos\frac{x}{2}-\sin\frac{x}{2}}{\cos\frac{x}{2}+\sin\frac{x}{2}-\cos\frac{x}{2}+\sin\frac{x}{2}}\right)$$ **(1)**

$$=\cot^{-1}\left(\frac{2\cos\frac{x}{2}}{2\sin\frac{x}{2}}\right)=\cot^{-1}\left(\cot\frac{x}{2}\right)$$ **(1/2)**

$\because$ The principal value branch of $\cot^{-1}x$ is $(0,\pi)$.

$\therefore$ $$\cot^{-1}\left(\cot\frac{x}{2}\right)=\frac{x}{2}$$

$$\left[\because x\in\left(0,\frac{\pi}{4}\right)\Rightarrow\frac{x}{2}\in\left(0,\frac{\pi}{8}\right)\right]$$ **(1)**

$\therefore$ LHS = RHS **Hence proved.**

NOTE If $x \in \left(0, \frac{\pi}{2}\right)$, then $\sqrt{1-\sin x} = \cos\frac{x}{2} - \sin\frac{x}{2}$

and if $x \in \left(\frac{\pi}{2}, \pi\right)$, then $\sqrt{1-\sin x} = \sin\frac{x}{2} - \cos\frac{x}{2}$

Alternate Method

$$\text{LHS} = \cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right)$$

$$= \cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}} \times \frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}+\sqrt{1-\sin x}}\right)$$

[by rationalising denominator] **(1)**

$$= \cot^{-1}\left[\frac{(\sqrt{1+\sin x}+\sqrt{1-\sin x})^2}{(\sqrt{1+\sin x})^2-(\sqrt{1-\sin x})^2}\right]$$

$[\because (a+b)(a-b) = a^2 - b^2]$

$$= \cot^{-1}\left(\frac{1+\sin x+1-\sin x+2\sqrt{1-\sin^2 x}}{1+\sin x-1+\sin x}\right)$$

$[\because (a+b)^2 = a^2+b^2+2ab]$

$$= \cot^{-1}\left(\frac{2+2\cos x}{2\sin x}\right) \quad [\because \cos x = \sqrt{1-\sin^2 x}] \ \textbf{(1)}$$

$$= \cot^{-1}\left(\frac{1+\cos x}{\sin x}\right) = \cot^{-1}\left(\frac{2\cos^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}\right) \quad \textbf{(1)}$$

$$\left[\because 1+\cos\theta = 2\cos^2\frac{\theta}{2} \text{ and } \sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right]$$

$$= \cot^{-1}\left[\cot\frac{x}{2}\right] = \frac{x}{2} = \text{RHS}$$

$[\because \cot^{-1}(\cot\theta) = \theta]$ **(1)**

42. Prove that $\sin^{-1}\left(\frac{8}{17}\right)+\sin^{-1}\left(\frac{3}{5}\right)=\cos^{-1}\left(\frac{36}{85}\right)$.

All India 2014C; Delhi 2012, 2010C

To prove, $\sin^{-1}\left(\frac{8}{17}\right)+\sin^{-1}\left(\frac{3}{5}\right)=\cos^{-1}\left(\frac{36}{85}\right)$

Let $\sin^{-1}\left(\frac{8}{17}\right)=x$ ...(i)

and $\sin^{-1}\left(\frac{3}{5}\right)=y$ ...(ii)

$\Rightarrow \quad \sin x=\frac{8}{17}$ and $\sin y=\frac{3}{5}$ **(1)**

Now, $\cos^2 x=1-\sin^2 x$

$$=1-\frac{64}{289}=\frac{225}{289} \Rightarrow \cos x=\sqrt{\frac{225}{289}}$$

$\Rightarrow \quad \cos x=\frac{15}{17}$ **(1)**

Also, $\cos^2 y=1-\sin^2 y=1-\frac{9}{25}$

$\Rightarrow \quad \cos y=\sqrt{\frac{16}{25}} \Rightarrow \cos y=\frac{4}{5}$ **(1)**

We know that,

$$\cos(x+y)=\cos x\cos y-\sin x\sin y$$

$$\Rightarrow \cos(x+y)=\left(\frac{15}{17}\times\frac{4}{5}\right)-\left(\frac{8}{17}\times\frac{3}{5}\right)$$

$$\Rightarrow \cos(x+y)=\frac{60}{85}-\frac{24}{85}=\frac{36}{85}$$

$\Rightarrow \quad x+y=\cos^{-1}\left(\frac{36}{85}\right)$ ...(iii)

$\Rightarrow \quad \sin^{-1}\frac{8}{17}+\sin^{-1}\frac{3}{5}=\cos^{-1}\frac{36}{85}$ **(1)**

[from Eqs. (i), (ii) and (iii)] **Hence proved.**

43. Show that $\tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right)=\frac{4-\sqrt{7}}{3}$. **All India 2013**

. To prove, $\tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right)=\frac{4-\sqrt{7}}{3}$

$$\text{LHS}=\tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right) \quad \ldots(\text{i})$$

Let $\frac{1}{2}\sin^{-1}\left(\frac{3}{4}\right)=\theta$ ...(ii)

$$\Rightarrow \quad \sin^{-1}\left(\frac{3}{4}\right)=2\theta \Rightarrow \sin 2\theta=\frac{3}{4} \quad \textbf{(1)}$$

$$\Rightarrow \quad \frac{2\tan\theta}{1+\tan^2\theta}=\frac{3}{4} \quad \left[\because \sin 2\theta=\frac{2\tan\theta}{1+\tan^2\theta}\right]$$

$$\Rightarrow \quad 8\tan\theta=3+3\tan^2\theta$$

$$\Rightarrow 3\tan^2\theta-8\tan\theta+3=0$$

Now, by Sridharacharya's rule

$$\tan\theta=\frac{8\pm\sqrt{64-36}}{2\times 3}=\frac{8\pm\sqrt{28}}{6} \quad \textbf{(1)}$$

$$\Rightarrow=\frac{8\pm 2\sqrt{7}}{6}=\frac{4\pm\sqrt{7}}{3} \Rightarrow \theta=\tan^{-1}\left(\frac{4\pm\sqrt{7}}{3}\right)$$

$$[\because \tan\theta=\phi \Rightarrow \theta=\tan^{-1}\phi]$$

So, from Eq. (ii), we get

$$\frac{1}{2}\sin^{-1}\left(\frac{3}{4}\right)=\tan^{-1}\left(\frac{4\pm\sqrt{7}}{3}\right) \quad \textbf{(1)}$$

Taking (–)ve sign, we get

$$\frac{1}{2}\sin^{-1}\left(\frac{3}{4}\right)=\tan^{-1}\left(\frac{4-\sqrt{7}}{3}\right)$$

On taking tan both sides, we get

$$\tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right)=\tan\left\{\tan^{-1}\left(\frac{4-\sqrt{7}}{3}\right)\right\}$$

$$\Rightarrow \tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right)=\frac{4-\sqrt{7}}{3}$$

$$[\because \tan(\tan^{-1}\theta)=\theta] \quad \textbf{(1)}$$

$=$ RHS **Hence proved.**

44. Prove that $\sin^{-1}\frac{8}{17}+\sin^{-1}\frac{3}{5}=\tan^{-1}\frac{77}{36}$.

Delhi 2013C

To prove, $\sin^{-1}\frac{8}{17}+\sin^{-1}\frac{3}{5}=\tan^{-1}\frac{77}{36}$

Let $\sin^{-1}\frac{8}{17}=x$ and $\sin^{-1}\frac{3}{5}=y$

$\Rightarrow \quad \sin x=\frac{8}{17}$ and $\sin y=\frac{3}{5}$ **(1)**

Now, $\cos^2 x=1-\sin^2 x=1-\frac{64}{289}=\frac{225}{289}$

$\Rightarrow \quad \cos x=\sqrt{\frac{225}{289}}=\frac{15}{17}$

Also, $\cos^2 y=1-\sin^2 y=1-\frac{9}{25}=\frac{16}{25}$

$\Rightarrow \quad \cos y=\sqrt{16/25}=4/5$ **(1)**

We know that,

$$\cos(x+y)=\cos x\cos y-\sin x\sin y$$

$$=\frac{15}{17}\times\frac{4}{5}-\frac{8}{17}\times\frac{3}{5}=\frac{60}{85}-\frac{24}{85}=\frac{36}{85}$$

$\Rightarrow \quad x+y=\cos^{-1}\left(\frac{36}{85}\right)$

$\Rightarrow \quad \sin^{-1}\left(\frac{8}{17}\right)+\sin^{-1}\left(\frac{3}{5}\right)=\cos^{-1}\left(\frac{36}{85}\right)$ ...(i)

Again, let $\cos^{-1}\left(\frac{36}{85}\right)=z \Rightarrow \cos z=\frac{36}{85}$

then, $\tan z=\frac{77}{36}$ **(1)**

$\Rightarrow \quad z=\tan^{-1}\frac{77}{36} \Rightarrow \cos^{-1}\frac{36}{85}=\tan^{-1}\frac{77}{36}$

On putting this value in Eq. (i), we get

$\sin^{-1}\left(\frac{8}{17}\right)+\sin^{-1}\left(\frac{3}{5}\right)=\tan^{-1}\left(\frac{77}{36}\right)$ **(1)**

Hence proved.

45. Solve for x, $\tan^{-1}3x + \tan^{-1}2x = \frac{\pi}{4}$.

Delhi 2013C, 2009 ; All India 2009C, 2008

Given equation is $\tan^{-1}3x + \tan^{-1}2x = \frac{\pi}{4}$

$$\Rightarrow \quad \tan^{-1}\left(\frac{3x+2x}{1-3x\times 2x}\right) = \frac{\pi}{4} \quad \textbf{(1)}$$

$$\left[\because \tan^{-1}A + \tan^{-1}B = \tan^{-1}\left(\frac{A+B}{1-AB}\right)\right]$$

$$\Rightarrow \tan^{-1}\left(\frac{5x}{1-6x^2}\right) = \frac{\pi}{4} \Rightarrow \frac{5x}{1-6x^2} = \tan\frac{\pi}{4}$$

$$[\because \tan^{-1}(\theta) = \phi \Rightarrow \theta = \tan\phi]$$

$$\Rightarrow \quad \frac{5x}{1-6x^2} = 1$$

$$\Rightarrow \quad 5x = 1 - 6x^2 \quad \textbf{(1)}$$

$$\Rightarrow \quad 6x^2 + 5x - 1 = 0$$

$$\Rightarrow \quad 6x^2 + 6x - x - 1 = 0$$

$$\Rightarrow \quad 6x(x+1) - 1(x+1) = 0 \quad \textbf{(1)}$$

$$\Rightarrow \quad (6x-1)(x+1) = 0$$

$$\Rightarrow \quad 6x - 1 = 0 \text{ or } x + 1 = 0$$

$$\Rightarrow \quad x = 1/6 \text{ or } x = -1 \quad \textbf{(1)}$$

But $x = -1$ does not satisfy the given equation.

Hence, required value of x is $\frac{1}{6}$.

46. Find the value of the following:

$$\tan\frac{1}{2}\left[\sin^{-1}\left(\frac{2x}{1+x^2}\right) + \cos^{-1}\left(\frac{1-y^2}{1+y^2}\right)\right],$$

if $|x| < 1$, $y > 0$ and $xy < 1$ **Delhi 2013**

Firstly, use the relation

$$2\tan^{-1}x=\sin^{-1}\left(\frac{2x}{1+x^2}\right)=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$ to

convert into $\tan^{-1}$, then use identity relation $\tan(\tan^{-1}\theta)=\theta$.

$$\tan\left\{\frac{1}{2}\sin^{-1}\left(\frac{2x}{1+x^2}\right)+\frac{1}{2}\cos^{-1}\left(\frac{1-y^2}{1+y^2}\right)\right\}$$

$$=\tan\left[\frac{1}{2}(2\tan^{-1}x)+\frac{1}{2}(2\tan^{-1}y)\right] \quad \textbf{(2)}$$

$$\left[\because 2\tan^{-1}x=\sin^{-1}\left(\frac{2x}{1+x^2}\right)=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)\right]$$

$$=\tan(\tan^{-1}x+\tan^{-1}y)$$

$$=\tan\left[\tan^{-1}\left(\frac{x+y}{1-xy}\right)\right]$$

$$\left[\because \tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right)\right]$$

$$=\frac{x+y}{1-xy} \qquad [\because \tan(\tan^{-1}\theta)=\theta] \quad \textbf{(2)}$$

47. Prove that

$$\tan^{-1}\left(\frac{1}{2}\right)+\tan^{-1}\left(\frac{1}{5}\right)+\tan^{-1}\left(\frac{1}{8}\right)=\frac{\pi}{4}.$$

Delhi 2013; All India 2011, 2008C

To prove,

$$\tan^{-1}\left(\frac{1}{2}\right)+\tan^{-1}\left(\frac{1}{5}\right)+\tan^{-1}\left(\frac{1}{8}\right)=\frac{\pi}{4}$$

$$\text{LHS}=\tan^{-1}\left(\frac{1}{2}\right)+\tan^{-1}\left(\frac{1}{5}\right)+\tan^{-1}\left(\frac{1}{8}\right)$$

$$=\tan^{-1}\left(\frac{\frac{1}{2}+\frac{1}{5}}{1-\frac{1}{10}}\right)+\tan^{-1}\left(\frac{1}{8}\right) \qquad \textbf{(1½)}$$

$$\left[\because \tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right), xy<1\right]$$

$$=\tan^{-1}\left(\frac{7}{9}\right)+\tan^{-1}\left(\frac{1}{8}\right)$$

$$=\tan^{-1}\left(\frac{\frac{7}{9}+\frac{1}{8}}{1-\frac{7}{72}}\right) \qquad \textbf{(1½)}$$

$$\left[\because \tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right), xy<1\right]$$

$$=\tan^{-1}\left(\frac{56+9}{72-7}\right)=\tan^{-1}\left(\frac{65}{65}\right)$$

$$=\tan^{-1}(1)=\tan^{-1}\left(\tan\frac{\pi}{4}\right)=\frac{\pi}{4}=\text{RHS} \qquad \textbf{(1)}$$

Hence proved.

48. Prove that

$$\tan^{-1}\left(\frac{\cos x}{1+\sin x}\right)=\frac{\pi}{4}-\frac{x}{2}, x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right).$$

Delhi 2012;HOTS

Firstly, use the relation $\cos\theta = \cos^2\frac{x}{2} - \sin^2\frac{x}{2}$ and $\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$. After that use the relation $\tan(A-B) = \frac{\tan A - \tan B}{1+\tan A \tan B}$ and simplify it.

$$\text{LHS} = \tan^{-1}\left(\frac{\cos x}{1+\sin x}\right)$$

$$= \tan^{-1}\left(\frac{\cos^2\frac{x}{2} - \sin^2\frac{x}{2}}{\cos^2\frac{x}{2} + \sin^2\frac{x}{2} + 2\sin\frac{x}{2}\cos\frac{x}{2}}\right)$$

$$\left[\because \cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2} \text{ and } \sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}\right] \quad \textbf{(1)}$$

$$= \tan^{-1}\left[\frac{\left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)}{\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)^2}\right]$$

$$[\because a^2 - b^2 = (a-b)(a+b) \text{ and } (a+b)^2 = a^2 + 2ab + b^2]$$

$$= \tan^{-1}\left(\frac{\cos\frac{x}{2} - \sin\frac{x}{2}}{\cos\frac{x}{2} + \sin\frac{x}{2}}\right) \quad \textbf{(1)}$$

On dividing the numerator and denominator by $\cos x/2$, we get

$$\text{LHS} = \tan^{-1}\left(\frac{\frac{\cos\frac{x}{2}}{\cos\frac{x}{2}} - \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}}{\frac{\cos\frac{x}{2}}{\cos\frac{x}{2}} + \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}}\right) = \tan^{-1}\left(\frac{1-\tan\frac{x}{2}}{1+\tan\frac{x}{2}}\right)$$

$$= \tan^{-1}\left(\frac{\tan\frac{\pi}{4} - \tan\frac{x}{2}}{1 + \tan\frac{\pi}{4}\cdot\tan\frac{x}{2}}\right)$$ **(1)**

$$\left[\because 1 = \tan\frac{\pi}{4} \text{ and } 1\cdot\tan\frac{x}{2} = \tan\frac{\pi}{4}\cdot\tan\frac{x}{2}\right]$$

$$= \tan^{-1}\left[\tan\left(\frac{\pi}{4} - \frac{x}{2}\right)\right]$$

$$\left[\because \frac{\tan A - \tan B}{1 + \tan A \tan B} = \tan(A - B)\right]$$

$$= \frac{\pi}{4} - \frac{x}{2}$$

$$\left[\because \tan^{-1}(\tan\theta) = \theta;\ x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right]$$

$= \text{RHS}$ **(1) Hence proved.**

49. Prove that

$$\cos^{-1}\left(\frac{4}{5}\right) + \cos^{-1}\left(\frac{12}{13}\right) = \cos^{-1}\left(\frac{33}{65}\right).$$

All India 2012; Delhi 2010C, 2009

To prove, $\cos^{-1}\left(\frac{4}{5}\right)+\cos^{-1}\left(\frac{12}{13}\right)=\cos^{-1}\left(\frac{33}{65}\right)$

Let $\cos^{-1}\left(\frac{4}{5}\right)=x$ ...(i)

and $\cos^{-1}\left(\frac{12}{13}\right)=y$ ...(ii)

$\Rightarrow \cos x=\frac{4}{5}$ and $\cos y=\frac{12}{13}$ **(1)**

We know that,

$$\sin^2 x=1-\cos^2 x=1-\frac{16}{25}=\frac{9}{25}$$

$$\Rightarrow \sin x=\sqrt{\frac{9}{25}}=\frac{3}{5}$$

and $\sin^2 y=1-\cos^2 y=1-\frac{144}{169}=\frac{25}{169}$

$\sin y=\sqrt{\frac{25}{169}} \Rightarrow =\frac{5}{13}$ **(1)**

Now, we know that,

$$\cos(x+y)=\cos x\cos y-\sin x\sin y$$

$$\Rightarrow \cos(x+y)=\left(\frac{4}{5}\times\frac{12}{13}\right)-\left(\frac{3}{5}\times\frac{5}{13}\right)$$

$$=\frac{48}{65}-\frac{15}{65}$$

$\Rightarrow \cos(x+y)=\frac{33}{65}$ **(1)**

$$\Rightarrow x+y=\cos^{-1}\frac{33}{65}$$

$\Rightarrow \cos^{-1}\frac{4}{5}+\cos^{-1}\frac{12}{13}=\cos^{-1}\frac{33}{65}$ ...(iii)

[From Eqs. (i), (ii) and (iii)] **(1) Hence proved.**

50. Prove that $\sin^{-1}\left(\frac{63}{65}\right)=\sin^{-1}\left(\frac{5}{13}\right)+\cos^{-1}\left(\frac{3}{5}\right)$.

Foreign 2012

. To prove,

$$\sin^{-1}\left(\frac{63}{65}\right) = \sin^{-1}\left(\frac{5}{13}\right) + \cos^{-1}\left(\frac{3}{5}\right)$$

$$\text{RHS} = \sin^{-1}\left(\frac{5}{13}\right) + \cos^{-1}\left(\frac{3}{5}\right)$$

Let $\sin^{-1}\frac{5}{13} = x \Rightarrow \sin x = \frac{5}{13}$ ...(i)

and $\cos^{-1}\frac{3}{5} = y \Rightarrow \cos y = \frac{3}{5}$ ...(ii) **(1)**

Also, $\cos x = \sqrt{1-\sin^2 x}$

$$= \sqrt{1-\frac{25}{169}} = \sqrt{\frac{144}{169}} = \frac{12}{13}$$

and $\sin y = \sqrt{1-\cos^2 y}$

$$= \sqrt{1-\frac{9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5}$$ **(1)**

We know that,

$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$

$$= \frac{5}{13}\times\frac{3}{5}+\frac{12}{13}\times\frac{4}{5} = \frac{15}{65}+\frac{48}{65} = \frac{63}{65}$$ **(1)**

$\Rightarrow \quad x + y = \sin^{-1}\left(\frac{63}{65}\right)$ ...(iii)

$[\because \sin\theta = \phi \Rightarrow \theta = \sin^{-1}\phi]$

$\Rightarrow \quad \sin^{-1}\left(\frac{5}{13}\right) + \cos^{-1}\left(\frac{3}{5}\right) = \sin^{-1}\left(\frac{63}{65}\right)$ **(1)**

[from Eqs. (i), (ii) and (iii)]

Hence proved.

51. Solve for x,

$$2\tan^{-1}(\sin x) = \tan^{-1}(2\sec x),\ x \neq \frac{\pi}{2}.$$

Foreign 2012

To solve, $2\tan^{-1}(\sin x) = \tan^{-1}(2\sec x)$

$$\Rightarrow \quad \tan^{-1}\left(\frac{2\sin x}{1-\sin^2 x}\right) = \tan^{-1}\left(\frac{2}{\cos x}\right) \qquad \textbf{(1½)}$$

$$\left[\because\ 2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right) \quad \frac{1}{\cos x}\right]$$

On comparing, we get

$$\Rightarrow \quad \frac{2\sin x}{\cos^2 x} = \frac{2}{\cos x} \qquad \textbf{(1)}$$

$$\Rightarrow \quad \tan x = 1 \qquad \textbf{(1/2)}$$

$$\Rightarrow \quad x = \tan^{-1}(1) = \tan^{-1}\left(\tan\frac{\pi}{4}\right) = \frac{\pi}{4} \qquad \textbf{(1)}$$

52. Find the value of $\tan^{-1}\left(\frac{x}{y}\right) - \tan^{-1}\left(\frac{x-y}{x+y}\right)$.

Delhi 2011

Use the relation

$$\tan^{-1} x - \tan^{-1} y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$$

We have, $\tan^{-1}\left(\frac{x}{y}\right) - \tan^{-1}\left(\frac{x-y}{x+y}\right)$

$$= \tan^{-1}\left(\frac{\frac{x}{y} - \frac{x-y}{x+y}}{1+\frac{x}{y}\cdot\frac{x-y}{x+y}}\right) \qquad \mathbf{(1½)}$$

$$\left[\because \tan^{-1}\theta - \tan^{-1}\phi = \tan^{-1}\left(\frac{\theta-\phi}{1+\theta\cdot\phi}\right)\right]$$

$$= \tan^{-1}\left[\frac{x(x+y) - y(x-y)}{y(x+y) + x(x-y)}\right]$$

$$= \tan^{-1}\left(\frac{x^2 + xy - xy + y^2}{xy + y^2 + x^2 - xy}\right) \qquad \mathbf{(1½)}$$

$$= \tan^{-1}\left(\frac{x^2+y^2}{y^2+x^2}\right) = \tan^{-1}(1)$$

$$\left[\because \text{principal value branch of} \tan^{-1} x \text{ is } \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)\right]$$

$$= \tan^{-1}\left(\tan\frac{\pi}{4}\right) = \frac{\pi}{4} \qquad \mathbf{(1)}$$

53. Prove that

$$2\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{7}\right) = \tan^{-1}\left(\frac{31}{17}\right).$$

All India 2011; Delhi 2009C, 2008C

Use the relation, $2\tan^{-1}x=\tan^{-1}\left(\frac{2x}{1-x^2}\right)$

and then $\tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right)$

$$\text{LHS}=2\tan^{-1}\left(\frac{1}{2}\right)+\tan^{-1}\left(\frac{1}{7}\right)$$

$$=\tan^{-1}\left[\frac{2\times(1/2)}{1-(1/2)^2}\right]+\tan^{-1}\frac{1}{7}$$

$$\left[\because\ 2\tan^{-1}x=\tan^{-1}\left(\frac{2x}{1-x^2}\right)\right]$$ **(1½)**

$$=\tan^{-1}\left(\frac{1}{1-\frac{1}{4}}\right)+\tan^{-1}\left(\frac{1}{7}\right)$$

$$=\tan^{-1}\left(\frac{1}{3/4}\right)+\tan^{-1}\left(\frac{1}{7}\right)$$

$$=\tan^{-1}\left(\frac{4}{3}\right)+\tan^{-1}\left(\frac{1}{7}\right)=\tan^{-1}\left(\frac{\frac{4}{3}+\frac{1}{7}}{1-\frac{4}{3}\times\frac{1}{7}}\right)$$ **(1½)**

$$\left[\because\ \tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right)\right]$$

$$=\tan^{-1}\left(\frac{\frac{28+3}{21}}{\frac{21-4}{21}}\right)=\tan^{-1}\frac{31}{17}=\text{RHS}$$ **(1)**

Hence proved.

54. Prove that $\frac{9\pi}{8}-\frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right)=\frac{9}{4}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)$.

Foreign 2011

Method I

Let $\sin^{-1}\left(\frac{1}{3}\right) = x$ and $\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right) = y$

Then, we get

$$\sin x = \frac{1}{3} \text{ and } \sin y = \frac{2\sqrt{2}}{3}$$

Now, $\cos^2 x = 1 - \sin^2 x = 1 - \frac{1}{9} = \frac{8}{9}$

$\Rightarrow \quad \cos x = \sqrt{\frac{8}{9}} = \frac{2\sqrt{2}}{3}$ **(1)**

Similarly, $\cos^2 y = 1 - \sin^2 y = 1 - \frac{8}{9} = \frac{1}{9}$

$\Rightarrow \quad \cos y = \sqrt{\frac{1}{9}} = \frac{1}{3}$ **(1)**

Now, $\sin(x + y) = \sin x \cos y + \cos x \sin y$

$$= \frac{1}{3} \times \frac{1}{3} + \frac{2\sqrt{2}}{3} \times \frac{2\sqrt{2}}{3}$$

$= \frac{1}{9} + \frac{8}{9} = \frac{9}{9} = 1$ **(1)**

$\Rightarrow \quad \sin(x + y) = 1 = \sin\frac{\pi}{2}$

$\left[\because \text{ principal value branch of } \sin^{-1} x \text{ is } \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\right]$

$$x + y = \frac{\pi}{2}$$

$$\Rightarrow \quad \sin^{-1}\left(\frac{1}{3}\right) + \sin^{-1}\left(\frac{2\sqrt{2}}{3}\right) = \frac{\pi}{2}$$

$$\left[\because x = \sin^{-1}\left(\frac{1}{3}\right) \text{ and } y = \sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)\right]$$

$$\Rightarrow \frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right) + \frac{9}{4}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right) = \frac{9\pi}{8}$$

[multiplying both sides by 9/4]

$$\Rightarrow \quad \frac{9\pi}{8} - \frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right) = \frac{9}{4}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right) \quad \textbf{(1)}$$

Hence proved.

***Method* II**

To prove that

$$\frac{9\pi}{8} - \frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right) = \frac{9}{4}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)$$

$$\text{LHS} = \frac{9\pi}{8} - \frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right)$$

$$= \frac{9}{4}\left[\frac{\pi}{2} - \sin^{-1}\left(\frac{1}{3}\right)\right] \quad \textbf{(1)}$$

$$= \frac{9}{4}\left[\cos^{-1}\left(\frac{1}{3}\right)\right] \quad \left[\because \cos^{-1}\theta = \frac{\pi}{2} - \sin^{-1}\theta\right]$$

(1)

$$= \frac{9}{4}\sin^{-1}\left(\sqrt{1 - \frac{1}{9}}\right)$$

$[\because \cos^{-1} x = \sin^{-1}\sqrt{1 - x^2}]$ **(1)**

$$= \frac{9}{4}\sin^{-1}\left(\sqrt{\frac{8}{9}}\right) \quad \textbf{(1)}$$

$= \text{RHS}$ **Hence proved.**

55. Prove that $\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{2}{9}=\frac{1}{2}\tan^{-1}\frac{4}{3}$.

All India 2011C

To prove,

$$\tan^{-1}\left(\frac{1}{4}\right)+\tan^{-1}\left(\frac{2}{9}\right)=\left(\frac{1}{2}\right)\tan^{-1}\left(\frac{4}{3}\right) \quad \ldots(i)$$

Above equation may be rewritten as

$$2\left[\tan^{-1}\left(\frac{1}{4}\right)+\tan^{-1}\left(\frac{2}{9}\right)\right]=\tan^{-1}\left(\frac{4}{3}\right) \quad \ldots(ii)$$

(1/2)

Now, $\text{LHS}=2\left[\tan^{-1}\left(\frac{1}{4}\right)+\tan^{-1}\left(\frac{2}{9}\right)\right]$

$$=2\left[\tan^{-1}\left(\frac{\frac{1}{4}+\frac{2}{9}}{1-\frac{1}{4}\times\frac{2}{9}}\right)\right] \quad \textbf{(1)}$$

$$\left[\because \tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right)\right]$$

$$=2\tan^{-1}\left(\frac{\frac{9+8}{36}}{\frac{36-2}{36}}\right)=2\tan^{-1}\left(\frac{17}{34}\right)$$

$$=2\tan^{-1}\left(\frac{1}{2}\right)=\tan^{-1}\left[\frac{2\times\frac{1}{2}}{1-\left(\frac{1}{2}\right)^2}\right] \quad \textbf{(1)}$$

$$\left[\because 2\tan^{-1}x=\tan^{-1}\left(\frac{2x}{1-x^2}\right)\right]$$

$$=\tan^{-1}\left(\frac{1}{1-\frac{1}{4}}\right)=\tan^{-1}\left(\frac{4}{3}\right)=\text{RHS} \quad \textbf{(1½)}$$

Hence proved.

56. Solve for x, $\cos(2\sin^{-1}x) = \frac{1}{9}, x > 0$.

All India 2011C; HOTS

Given equation is

$$\cos(2\sin^{-1}x) = \frac{1}{9}, x > 0 \quad \ldots\text{(i)}$$

Put $\sin^{-1}x = y \Rightarrow x = \sin y$ **(1/2)**

Then, Eq. (i) becomes, $\cos 2y = \frac{1}{9}$

$[\because \sin y = x]$

$$\Rightarrow 1 - 2\sin^2 y = \frac{1}{9}$$

$[\because \cos 2\theta = 1 - 2\sin^2\theta]$ **(1)**

$$\Rightarrow 2\sin^2 y = 1 - \frac{1}{9} = \frac{8}{9}$$ **(1/2)**

$$\Rightarrow \sin^2 y = \frac{4}{9} \Rightarrow x^2 = \frac{4}{9} \quad [\because \sin y = x]$$

$$\Rightarrow x = \pm\frac{2}{3}$$ [taking square root] **(1)**

But it is given that, $x > 0$

$$\therefore x = \frac{2}{3}$$ **(1)**

Alternate Method

Given equation is $\cos(2\sin^{-1}x) = \frac{1}{9}, x > 0$

$$\Rightarrow \cos(\sin^{-1}2x\sqrt{1-x^2}) = \frac{1}{9}$$

$\left[\because 2\sin^{-1}x = \sin^{-1}(2x\sqrt{1-x^2})\right]$ **(1)**

$$\Rightarrow \cos\left[\cos^{-1}\sqrt{1-(2x\sqrt{1-x^2})^2}\right] = \frac{1}{9}$$

$[\because \sin^{-1}x = \cos^{-1}\sqrt{1-x^2}]$ **(1)**

$$\Rightarrow \sqrt{1-4x^2(1-x^2)} = \frac{1}{9} \quad [\because \cos(\cos^{-1}\theta) = \theta]$$

On squaring both sides, we get

$$81(1-4x^2+4x^4)=1$$

$$\Rightarrow \quad 324x^4-324x^2+80=0$$

$$\Rightarrow \quad 81x^4-81x^2+20=0$$

[dividing both sides by 4]

$$\Rightarrow \quad 81x^4-45x^2-36x^2+20=0$$

$$\Rightarrow \quad 9x^2(9x^2-5)-4(9x^2-5)=0$$

$$\Rightarrow \quad (9x^2-5)(9x^2-4)=0$$

$$\Rightarrow \quad x^2=\frac{5}{9} \text{ or } \frac{4}{9} \Rightarrow x=\pm\frac{\sqrt{5}}{3} \text{ or } \pm\frac{2}{3}$$

But $x>0$

$\therefore \quad x=+\frac{\sqrt{5}}{3}$ or $\frac{2}{3}$ **(1)**

But here, $x=\frac{\sqrt{5}}{3}$ does not satisfy the given equation.

$\therefore \quad x=2/3$ is the only solution. **(1)**

NOTE While solving an equation, please be careful on squaring the equation. Sometimes, it may give extra value, which do not satisfy the given equation.

57. Prove that $2\tan^{-1}\frac{3}{4}-\tan^{-1}\frac{17}{31}=\frac{\pi}{4}$.

Delhi 2011C

Firstly, apply $2\tan^{-1}x=\tan^{-1}\left(\frac{2x}{1-x^2}\right)$ to evaluate $2\tan^{-1}\left(\frac{3}{4}\right)$ and then apply $\tan^{-1}x-\tan^{-1}y=\tan^{-1}\left(\frac{x-y}{1+xy}\right)$ and get the desired result.

To prove that $2\tan^{-1}\left(\frac{3}{4}\right)-\tan^{-1}\left(\frac{17}{31}\right)=\frac{\pi}{4}$

$$\text{LHS}=2\tan^{-1}\left(\frac{3}{4}\right)-\tan^{-1}\left(\frac{17}{31}\right)$$

$$=\tan^{-1}\left(\frac{2\times\frac{3}{4}}{1-\frac{9}{16}}\right)-\tan^{-1}\left(\frac{17}{31}\right) \qquad \textbf{(1)}$$

$$\left[\because 2\tan^{-1}x=\tan^{-1}\left(\frac{2x}{1-x^2}\right)\right]$$

$$=\tan^{-1}\left(\frac{3/2}{7/16}\right)-\tan^{-1}\left(\frac{17}{31}\right)$$

$$=\tan^{-1}\left(\frac{24}{7}\right)-\tan^{-1}\left(\frac{17}{31}\right)$$

$$=\tan^{-1}\left(\frac{\frac{24}{7}-\frac{17}{31}}{1+\frac{24}{7}\times\frac{17}{31}}\right) \qquad \textbf{(1)}$$

$$\left[\because \tan^{-1}x-\tan^{-1}y=\tan^{-1}\left(\frac{x-y}{1+xy}\right)\right]$$

$$=\tan^{-1}\left(\frac{24\times 31-17\times 7}{7\times 31+24\times 17}\right)$$

$$=\tan^{-1}\left(\frac{744-119}{217+408}\right)=\tan^{-1}\left(\frac{625}{625}\right)$$

$$=\tan^{-1}(1)$$

$$=\tan^{-1}\left(\tan\frac{\pi}{4}\right) \qquad \left[\because 1=\tan\frac{\pi}{4}\right]$$

$$=\frac{\pi}{4}=\text{RHS}$$

$\left[\because \text{principal value branch of } \tan^{-1}x \text{ is} \left(-\frac{\pi}{2},\frac{\pi}{2}\right).\right]$ **(2)**

58. Solve for x,

$$\tan^{-1}\left(\frac{2x}{1-x^2}\right)+\cot^{-1}\left(\frac{1-x^2}{2x}\right)=\frac{\pi}{3},$$

$-1<x<1.$

Delhi 2011C; HOTS

Firstly, write $\cot^{-1}\left(\frac{1-x^2}{2x}\right)=\tan^{-1}\left(\frac{2x}{1-x^2}\right)$ by applying formula $\cot^{-1}x=\tan^{-1}\frac{1}{x}$ and then proceed further.

Given equation is

$$\tan^{-1}\left(\frac{2x}{1-x^2}\right)+\cot^{-1}\left(\frac{1-x^2}{2x}\right)=\frac{\pi}{3}, -1<x<1$$

We know that, $\cot^{-1}x=\tan^{-1}\frac{1}{x}$, so by using this result, we may write

$$\cot^{-1}\left(\frac{1-x^2}{2x}\right)=\tan^{-1}\left(\frac{2x}{1-x^2}\right) \qquad \textbf{(1/2)}$$

Then, given equation becomes

$$\tan^{-1}\left(\frac{2x}{1-x^2}\right)+\tan^{-1}\left(\frac{2x}{1-x^2}\right)=\frac{\pi}{3} \qquad \textbf{(1/2)}$$

$$\Rightarrow \quad 2\tan^{-1}\left(\frac{2x}{1-x^2}\right)=\frac{\pi}{3} \qquad \textbf{(1/2)}$$

$$\Rightarrow \quad \tan^{-1}\left(\frac{2x}{1-x^2}\right)=\frac{\pi}{6}$$

$$\Rightarrow \quad \frac{2x}{1-x^2}=\tan\frac{\pi}{6}=\frac{1}{\sqrt{3}}$$

$$\Rightarrow \quad 2\sqrt{3}x=1-x^2 \qquad \textbf{(1)}$$

$$\Rightarrow \quad x^2+2\sqrt{3}x-1=0$$

$$\Rightarrow \quad x=\frac{-2\sqrt{3}\pm\sqrt{12+4}}{2}$$

$\left[\because \text{ we know that, } x=\frac{-b\pm\sqrt{D}}{2a} \text{ where, } D=b^2-4ac\right]$

$$\Rightarrow x=\frac{-2\sqrt{3}\pm 4}{2}=\frac{4-2\sqrt{3}}{2} \text{ or } \frac{-4-2\sqrt{3}}{2} \qquad \textbf{(1)}$$

$$\Rightarrow x=2-\sqrt{3} \text{ or } -(2+\sqrt{3})$$

But it is given that $-1<x<1$, so $x=-(2+\sqrt{3})$ is rejected.Hence, $x=2-\sqrt{3}$ **(1/2)**

59. Prove that

$$\tan^{-1}\sqrt{x} = \frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right), x \in (0, 1).$$

Delhi 2010;HOTS

> Put $\sqrt{x} = \tan\theta \Rightarrow \theta = \tan^{-1}\sqrt{x}$
>
> and then use $\cos 2\theta = \dfrac{1-\tan^2\theta}{1+\tan^2\theta}$

To prove, $\tan^{-1}\sqrt{x} = \frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right), x \in (0, 1)$

$$\text{RHS} = \frac{1}{2}\cos^{-1}\left[\frac{1-(\sqrt{x})^2}{1+(\sqrt{x})^2}\right] \quad \textbf{(1)}$$

On substituting $\sqrt{x} = \tan\theta$, we get

$$\text{RHS} = \frac{1}{2}\cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right) \quad \textbf{(1)}$$

$$= \frac{1}{2}\cos^{-1}(\cos 2\theta) \quad \textbf{(1)}$$

$$\left[\because \frac{1-\tan^2 A}{1+\tan^2 A} = \cos 2A\right]$$

$$= \frac{1}{2}(2\theta) = \theta \qquad [\because \cos^{-1}(\cos\theta) = \theta]$$

$$= \tan^{-1}\sqrt{x} \qquad [\because \theta = \tan^{-1}\sqrt{x}] \ \textbf{(1)}$$

$= \text{LHS}$ **Hence proved.**

Alternate Method

To prove, $\tan^{-1}\sqrt{x} = \frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right), x \in (0,1)$

$$\text{LHS} = \tan^{-1}\sqrt{x} = \frac{1}{2}(2\tan^{-1}\sqrt{x})$$

$$= \frac{1}{2} \times \cos^{-1}\left[\frac{1-(\sqrt{x})^2}{1+(\sqrt{x})^2}\right] \quad \textbf{(2)}$$

$$\left[\because 2\tan^{-1}x = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)\right]$$

$$= \frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right) = \text{RHS} \quad \textbf{(2)}$$

Hence proved.

60. Prove that

$$\cos^{-1}\left(\frac{12}{13}\right) + \sin^{-1}\left(\frac{3}{5}\right) = \sin^{-1}\left(\frac{56}{65}\right).$$ **Delhi 2010**

Use the relation $\sin^2 x + \cos^2 x = 1$. Find the value of $\sin x, \cos x, \sin y$ and $\cos y$, then use relation $\sin(x+y) = \sin x \cos y + \cos x \sin y$

Let $\cos^{-1}\frac{12}{13} = x$ and $\sin^{-1}\frac{3}{5} = y$

$\Rightarrow \quad \cos x = \frac{12}{13}$ and $\sin y = \frac{3}{5}$ **(1)**

$\therefore \quad \sin^2 x = 1 - \cos^2 x$

$$= 1 - \left(\frac{12}{13}\right)^2 \quad [\because \sin^2\theta + \cos^2\theta = 1]$$

$$= 1 - \frac{144}{169} = \frac{25}{169} \Rightarrow \sin x = \sqrt{\frac{25}{169}} = \frac{5}{13}$$

and $\cos^2 y = 1 - \sin^2 y = 1 - \left(\frac{3}{5}\right)^2 = 1 - \frac{9}{25} = \frac{16}{25}$

$\Rightarrow \quad \cos y = \sqrt{\frac{16}{25}} = \frac{4}{5}$ **(1)**

Now, $\sin(x+y) = \sin x \cos y + \cos x \sin y$

$$\Rightarrow \sin(x+y) = \left(\frac{5}{13} \times \frac{4}{5}\right) + \left(\frac{12}{13} \times \frac{3}{5}\right)$$

$$= \frac{20}{65} + \frac{36}{65} = \frac{56}{65}$$ **(1)**

$$\Rightarrow \sin(x+y) = \frac{56}{65} \Rightarrow x + y = \sin^{-1}\left(\frac{56}{65}\right)$$

$$\Rightarrow \cos^{-1}\left(\frac{12}{13}\right) + \sin^{-1}\left(\frac{3}{5}\right) = \sin^{-1}\left(\frac{56}{65}\right)$$

$$\left[\because x = \cos^{-1}\left(\frac{12}{13}\right) \text{ and } y = \sin^{-1}\left(\frac{3}{5}\right)\right]$$ **(1)**

Hence proved.

61. Prove that

$$\tan^{-1} x + \tan^{-1}\left(\frac{2x}{1-x^2}\right) = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right).$$

All India 2010

$= 3\theta = 3\tan^{-1} x \qquad [\because \theta = \tan^{-1} x]$ **(1)**

$= \tan^{-1} x + 2\tan^{-1} x$

$= \tan^{-1} x + \tan^{-1}\left(\frac{2x}{1-x^2}\right)$ **(1)**

$\left[\because 2\tan^{-1} x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)\right] = \text{LHS}$

Hence proved.

62. Prove that $\cos[\tan^{-1}\{\sin(\cot^{-1} x)\}] = \sqrt{\frac{1+x^2}{2+x^2}}$.

All India 2010

To prove $\cos[\tan^{-1}\{\sin(\cot^{-1} x)\}] = \sqrt{\frac{1+x^2}{2+x^2}}$

$\text{LHS} = \cos[\tan^{-1}\{\sin(\cot^{-1} x)\}]$

Let $\cot^{-1} x = \theta \Rightarrow x = \cot\theta$ **(1/2)**

Then, $\text{LHS} = \cos[\tan^{-1}(\sin\theta)]$

$= \cos\left[\tan^{-1}\left(\frac{1}{\operatorname{cosec}\theta}\right)\right]$ **(1/2)**

$\left[\because \operatorname{cosec}\theta = \frac{1}{\sin\theta}\right]$

$= \cos\left[\tan^{-1}\left(\frac{1}{\sqrt{1+\cot^2\theta}}\right)\right]$

$[\because \operatorname{cosec}^2\theta = 1 + \cot^2\theta]$

$= \cos\left[\tan^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)\right] \qquad [\because \cot\theta = x]$

$= \cos\phi$ **(1)**

where, $\tan^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right) = \phi$ or $\tan\phi = \frac{1}{\sqrt{1+x^2}}$

$$= \frac{1}{\sec\phi} \qquad \left[\because \sec\theta = \frac{1}{\cos\theta}\right]$$

$$= \frac{1}{\sqrt{1+\tan^2\phi}} \qquad [\because \tan^2\theta + 1 = \sec^2\theta]$$

$$= \frac{1}{\sqrt{1+\frac{1}{1+x^2}}} \qquad \left[\because \tan\phi = \frac{1}{\sqrt{1+x^2}}\right] \textbf{(1)}$$

$$= \frac{1}{\sqrt{\frac{1+x^2+1}{1+x^2}}} = \sqrt{\frac{1+x^2}{2+x^2}} = \text{RHS} \qquad \textbf{(1)}$$

Hence proved.

63. Solve for x, $\cos^{-1}x + \sin^{-1}\left(\frac{x}{2}\right) = \frac{\pi}{6}$.

All India 2010C

Given, $\cos^{-1}x + \sin^{-1}\left(\frac{x}{2}\right) = \frac{\pi}{6}$

$$\Rightarrow \quad \cos^{-1}x = \frac{\pi}{6} - \sin^{-1}\frac{x}{2}$$

$$\Rightarrow \quad x = \cos\left(\frac{\pi}{6} - \sin^{-1}\frac{x}{2}\right)$$

$$\Rightarrow \quad x = \cos\frac{\pi}{6}\cos\left(\sin^{-1}\frac{x}{2}\right) + \sin\frac{\pi}{6}\sin\left(\sin^{-1}\frac{x}{2}\right) \textbf{(1)}$$

$[\because \cos(x-y) = \cos x \cos y + \sin x \sin y]$

$$\Rightarrow \quad x = \frac{\sqrt{3}}{2}\cos\left(\sin^{-1}\frac{x}{2}\right) + \frac{1}{2}\cdot\frac{x}{2}$$

$[\because \sin(\sin^{-1}\theta) = \theta]$

$$\Rightarrow \quad x = \frac{\sqrt{3}}{2}\cos\left(\cos^{-1}\sqrt{1-\frac{x^2}{4}}\right) + \frac{x}{4}$$

2 (√ 4) 4

$$\left[\because \sin^{-1}\theta = \cos^{-1}\sqrt{1-\theta^2}\right]$$

$$\Rightarrow \quad x = \frac{\sqrt{3}}{2}\left(\sqrt{1-\frac{x^2}{4}}\right) + \frac{x}{4}$$

$$\Rightarrow \quad x - \frac{x}{4} = \frac{\sqrt{3}}{2}\left(\sqrt{1-\frac{x^2}{4}}\right)$$

$$\Rightarrow \quad \frac{3x}{4} = \frac{\sqrt{3}}{2}\left(\sqrt{1-\frac{x^2}{4}}\right) \qquad \textbf{(1)}$$

On squaring both sides, we get

$$\frac{9x^2}{16} = \frac{3}{4}\left(1-\frac{x^2}{4}\right) \Rightarrow \frac{3}{4}x^2 = 1 - \frac{x^2}{4}$$

$$\Rightarrow \quad \frac{3}{4}x^2 + \frac{x^2}{4} = 1 \Rightarrow \frac{4x^2}{4} = 1$$

$$\Rightarrow \quad x^2 = 1 \Rightarrow x = \pm 1 \qquad \textbf{(1)}$$

But $x = -1$, do not satisfy the given equation.

Hence, $x = 1$. **(1)**

64. Prove that $2\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7} = \frac{\pi}{4}$.

All India 2010C

To prove $2\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{7}\right) = \frac{\pi}{4}$

$$\text{LHS} = 2\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{7}\right) \qquad \ldots\text{(i)}$$

We know that, $2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$

Using this identity, we can write

$$2\tan^{-1}\left(\frac{1}{3}\right)=\tan^{-1}\left[\frac{2\times\frac{1}{3}}{1-\left(\frac{1}{3}\right)^2}\right]=\tan^{-1}\left(\frac{2/3}{1-\frac{1}{9}}\right)$$

$$\Rightarrow\ 2\tan^{-1}\left(\frac{1}{3}\right)=\tan^{-1}\left(\frac{3}{4}\right)$$ **(1½)**

On putting the value of $2\tan^{-1}\left(\frac{1}{3}\right)$ in Eq.(i), we get

$$\text{LHS}=\tan^{-1}\left(\frac{3}{4}\right)+\tan^{-1}\left(\frac{1}{7}\right)$$

$$=\tan^{-1}\left(\frac{\frac{3}{4}+\frac{1}{7}}{1-\frac{3}{4}\times\frac{1}{7}}\right)=\tan^{-1}\left(\frac{\frac{21+4}{28}}{\frac{28-3}{28}}\right)$$

$$\left[\because \tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right)\right]$$

$$=\tan^{-1}\left(\frac{\frac{25}{28}}{\frac{25}{28}}\right)=\tan^{-1}(1)$$ **(1½)**

$\because$ The principal value branch of $\tan^{-1}x$ is $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$.

$$\therefore\ \text{LHS}=\tan^{-1}\left(\tan\frac{\pi}{4}\right)=\frac{\pi}{4}=\text{RHS}$$ **(1)**

Hence proved.

65. Solve for x,

$\tan^{-1}\frac{x}{2}+\tan^{-1}\frac{x}{3}=\frac{\pi}{4},\ \sqrt{6}>x>0.$ **Delhi 2010C**

Given, $\tan^{-1}\frac{x}{2}+\tan^{-1}\frac{x}{3}=\frac{\pi}{4}, \sqrt{6}>x>0$

$$\Rightarrow \quad \tan^{-1}\left(\frac{\frac{x}{2}+\frac{x}{3}}{1-\frac{x^2}{6}}\right)=\frac{\pi}{4}$$

$$\left[\because \tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right)\right] \quad \mathbf{(1½)}$$

$$\Rightarrow \frac{\frac{3x+2x}{6}}{\frac{6-x^2}{6}}=\tan\frac{\pi}{4} \Rightarrow \frac{5x}{6-x^2}=1\left[\because \tan\frac{\pi}{4}=1\right]$$

$$\Rightarrow \quad 5x=6-x^2 \Rightarrow x^2+5x-6=0$$

$$\Rightarrow \quad x^2+6x-x-6=0$$

$$\Rightarrow \quad x(x+6)-1(x+6)=0$$

$$\Rightarrow \quad (x-1)(x+6)=0$$

$$\Rightarrow \quad x=1 \text{ or } -6 \quad \mathbf{(1½)}$$

But it is given that, $\sqrt{6}>x>0 \Rightarrow x>0$

$\therefore x=-6$ is rejected. Hence, $x=1$ **(1)**

66. Solve for x,

$$\tan^{-1}(x+2)+\tan^{-1}(x-2)=\tan^{-1}\left(\frac{8}{79}\right), x>0.$$

Delhi 2010C

Apply $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$ in LHS of given equation and then proceed further to obtain the desired result.

Given equation is

$$\tan^{-1}(x+2) + \tan^{-1}(x-2) = \tan^{-1}\left(\frac{8}{79}\right), x > 0$$

$$\Rightarrow \tan^{-1}\left[\frac{(x+2)+(x-2)}{1-(x+2)(x-2)}\right] = \tan^{-1}\left(\frac{8}{79}\right) \quad \mathbf{(1½)}$$

$$\left[\because \tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)\right]$$

$$\Rightarrow \left[\frac{2x}{1-(x^2-4)}\right] = \frac{8}{79} \quad \mathbf{(1/2)}$$

$$[\because (a+b)(a-b) = a^2 - b^2]$$

$$\Rightarrow \frac{2x}{5-x^2} = \frac{8}{79} \Rightarrow \frac{x}{5-x^2} = \frac{4}{79}$$

$$\Rightarrow 79x = 20 - 4x^2$$

$$\Rightarrow 4x^2 + 79x - 20 = 0$$

$$\Rightarrow 4x^2 + 80x - x - 20 = 0$$

$$\Rightarrow 4x(x+20) - 1(x+20) = 0$$

$$\Rightarrow (4x-1)(x+20) = 0$$

$$\Rightarrow x = \frac{1}{4} \text{ or } -20 \quad \mathbf{(1)}$$

But it is given that, $x > 0$

$\therefore$ $x = -20$ is rejected.

Hence, $x = 1/4$ **(1)**

67. Prove that

$$\sin^{-1}\left(\frac{4}{5}\right) + \sin^{-1}\left(\frac{5}{13}\right) + \sin^{-1}\left(\frac{16}{65}\right) = \frac{\pi}{2}.$$

Delhi 2009

To prove $\sin^{-1}\left(\frac{4}{5}\right)+\sin^{-1}\left(\frac{5}{13}\right)+\sin^{-1}\left(\frac{16}{65}\right)=\frac{\pi}{2}$

$$\text{LHS} = \sin^{-1}\left(\frac{4}{5}\right)+\sin^{-1}\left(\frac{5}{13}\right)+\sin^{-1}\left(\frac{16}{65}\right)$$

$$= \sin^{-1}\left(\frac{4}{5}\sqrt{1-\frac{25}{169}}+\frac{5}{13}\sqrt{1-\frac{16}{25}}\right)+\sin^{-1}\left(\frac{16}{65}\right)$$

$[\because \sin^{-1}x+\sin^{-1}y=\sin^{-1}(x\sqrt{1-y^2}+y\sqrt{1-x^2})]$ **68**

(1)

$$= \sin^{-1}\left(\frac{4}{5}\times\sqrt{\frac{144}{169}}+\frac{5}{13}\times\sqrt{\frac{9}{25}}\right)+\sin^{-1}\left(\frac{16}{65}\right)$$

$$= \sin^{-1}\left[\left(\frac{4}{5} \times \frac{12}{13}\right) + \left(\frac{5}{13} \times \frac{3}{5}\right)\right] + \sin^{-1}\left(\frac{16}{65}\right)$$

$$= \sin^{-1}\left(\frac{48}{65} + \frac{15}{65}\right) + \sin^{-1}\left(\frac{16}{65}\right)$$

$$= \sin^{-1}\left(\frac{63}{65}\right) + \sin^{-1}\left(\frac{16}{65}\right) \quad (1)$$

$$= \sin^{-1}\left[\frac{63}{65}\sqrt{1-\left(\frac{16}{65}\right)^2} + \frac{16}{65}\sqrt{1-\left(\frac{63}{65}\right)^2}\right]$$

$[\because \sin^{-1}x + \sin^{-1}y = \sin^{-1}(x\sqrt{1-y^2} + y\sqrt{1-x^2})]$

$$= \sin^{-1}\left(\frac{63}{65}\sqrt{\frac{4225-256}{4225}} + \frac{16}{65}\sqrt{\frac{4225-3969}{4225}}\right)$$

$$= \sin^{-1}\left(\frac{63}{65} \times \sqrt{\frac{3969}{4225}} + \frac{16}{65} \times \sqrt{\frac{256}{4225}}\right)$$

$$= \sin^{-1}\left(\frac{63}{65} \times \frac{63}{65} + \frac{16}{65} \times \frac{16}{65}\right)$$

$$= \sin^{-1}\left(\frac{3969+256}{4225}\right) = \sin^{-1}\left(\frac{4225}{4225}\right) = \sin^{-1}(1)$$

$\because$ The principal value branch of $\sin^{-1}x$ is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

$\therefore$ LHS $= \sin^{-1}\left(\sin\frac{\pi}{2}\right) = \frac{\pi}{2} =$ RHS **(1)**

Hence proved.

Alternate Method Given equation,

$$\sin^{-1}\left(\frac{4}{5}\right) + \sin^{-1}\left(\frac{5}{13}\right) + \sin^{-1}\left(\frac{16}{65}\right) = \frac{\pi}{2} \quad \ldots\text{(i)}$$

$$\Rightarrow \sin^{-1}\left(\frac{4}{5}\right) + \sin^{-1}\left(\frac{5}{13}\right) = \frac{\pi}{2} - \sin^{-1}\left(\frac{16}{65}\right)$$

$$\Rightarrow \sin^{-1}\left(\frac{4}{5}\right) + \sin^{-1}\left(\frac{5}{13}\right) = \cos^{-1}\left(\frac{16}{65}\right) \quad \ldots\text{(ii)}$$

$$\left[\because \frac{\pi}{2} - \sin^{-1}\theta = \cos^{-1}\theta\right] \quad \textbf{(1)}$$

Hence, Eqs. (i) and (ii) are equivalent. Now, we have to prove Eq. (ii).

Now do same as Que 42.

68. Prove that

$$\tan^{-1}\left(\frac{3}{4}\right)+\tan^{-1}\left(\frac{3}{5}\right)-\tan^{-1}\left(\frac{8}{19}\right)=\frac{\pi}{4}.$$

All India 2009C

. To prove, $\tan^{-1}\left(\frac{3}{4}\right)+\tan^{-1}\left(\frac{3}{5}\right)-\tan^{-1}\left(\frac{8}{19}\right)=\frac{\pi}{4}$

$$\text{LHS}=\left(\tan^{-1}\frac{3}{4}+\tan^{-1}\frac{3}{5}\right)-\tan^{-1}\frac{8}{19}$$

$$=\tan^{-1}\left(\frac{\frac{3}{4}+\frac{3}{5}}{1-\frac{9}{20}}\right)-\tan^{-1}\frac{8}{19}$$

$$\left[\because \tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right)\right] \quad \textbf{(1)}$$

$$=\tan^{-1}\left(\frac{27/20}{11/20}\right)-\tan^{-1}\frac{8}{19}$$

$$=\tan^{-1}\left(\frac{27}{11}\right)-\tan^{-1}\left(\frac{8}{19}\right) \quad \textbf{(1/2)}$$

$$=\tan^{-1}\left(\frac{\frac{27}{11}-\frac{8}{19}}{1+\frac{27}{11}\times\frac{8}{19}}\right) \quad \textbf{(1)}$$

$$=\tan^{-1}\left(\frac{\frac{513-88}{209}}{\frac{209+216}{209}}\right)=\tan^{-1}\left(\frac{425}{209}\times\frac{209}{425}\right) \quad \textbf{(1/2)}$$

$$=\tan^{-1}(1)$$

$\because$ The principal value branch of $\tan^{-1}x$ is $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$.

$$\therefore \quad \text{LHS}=\tan^{-1}\left[\tan\left(\frac{\pi}{4}\right)\right]=\frac{\pi}{4} \quad \textbf{(1)}$$

$$=\text{RHS}$$

Hence proved.

69. Prove that

$$\tan^{-1}\left(\frac{1}{3}\right)+\tan^{-1}\left(\frac{1}{5}\right)+\tan^{-1}\left(\frac{1}{7}\right)+\tan^{-1}\left(\frac{1}{8}\right)=\frac{\pi}{4}.$$

All India 2009C; Delhi 2008, 2008C

> Apply the identity,
>
> $\tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right)$ in first two terms and the last two terms of LHS and then apply the same identity again to the get the RHS.

To prove $\tan^{-1}\left(\frac{1}{3}\right)+\tan^{-1}\left(\frac{1}{5}\right)+\tan^{-1}\left(\frac{1}{7}\right)+\tan^{-1}\left(\frac{1}{8}\right)=\frac{\pi}{4}$

$$\text{LHS}=\left[\tan^{-1}\left(\frac{1}{3}\right)+\tan^{-1}\left(\frac{1}{5}\right)\right]+\left[\tan^{-1}\left(\frac{1}{7}\right)+\tan^{-1}\left(\frac{1}{8}\right)\right]$$

On applying the identity

$\tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right)$, we get

$$\text{LHS} = \tan^{-1}\left(\frac{\frac{1}{3}+\frac{1}{5}}{1-\frac{1}{15}}\right) + \tan^{-1}\left(\frac{\frac{1}{7}+\frac{1}{8}}{1-\frac{1}{56}}\right)$$ **(1½)**

$$= \tan^{-1}\left(\frac{8/15}{14/15}\right) + \tan^{-1}\left(\frac{15/56}{55/56}\right)$$

$$= \tan^{-1}\left(\frac{4}{7}\right) + \tan^{-1}\left(\frac{3}{11}\right)$$

$$= \tan^{-1}\left(\frac{\frac{4}{7}+\frac{3}{11}}{1-\frac{4}{7}\times\frac{3}{11}}\right) = \tan^{-1}\left(\frac{\frac{44+21}{77}}{\frac{77-12}{77}}\right)$$ **(1)**

$$= \tan^{-1}\left[\frac{65/77}{65/77}\right] = \tan^{-1}(1)$$ **(1)**

$\because$ The principal value branch of $\tan^{-1} x$ is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

$\therefore \text{LHS} = \tan^{-1}\left[\tan\left(\frac{\pi}{4}\right)\right] = \frac{\pi}{4} = \text{RHS}$ **(1/2)**

70. Solve for x,

$$\tan^{-1}\left(\frac{x-1}{x-2}\right) + \tan^{-1}\left(\frac{x+1}{x+2}\right) = \frac{\pi}{4}.$$ Delhi 2009C

Do same as Que 38. [**Ans.** $\pm 1/2$]

71. Solve for x,

$$\tan^{-1}\left(\frac{1+x}{1-x}\right) = \frac{\pi}{4} + \tan^{-1} x, 0 < x < 1.$$

Delhi 2008C

The given equation is

$$\tan^{-1}\left(\frac{1+x}{1-x}\right) = \frac{\pi}{4} + \tan^{-1} x$$

$$\Rightarrow \quad \tan^{-1}\left(\frac{1+x}{1-x}\right) - \tan^{-1} x = \frac{\pi}{4} \qquad \textbf{(1/2)}$$

$$\Rightarrow \quad \tan^{-1}\left[\frac{\frac{1+x}{1-x} - x}{1+\left(\frac{1+x}{1-x}\right)\cdot x}\right] = \frac{\pi}{4}$$

$$\left[\because \tan^{-1} x - \tan^{-1} y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)\right] \textbf{(1½)}$$

$$\Rightarrow \quad \frac{1+x-x+x^2}{1-x+x+x^2} = \tan\frac{\pi}{4} \qquad \textbf{(1)}$$

$$\Rightarrow \quad \frac{1+x^2}{1+x^2} = 1 \Rightarrow 1 = 1 \qquad \left[\because \tan\frac{\pi}{4} = 1\right]$$

Hence, the given equation has many solutions. **(1)**